Notes on the energy concept in supersymmetry

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Abstract

It is said everywhere that "energy is always positive" in supersymmetry, but this naive sentence could be confusing for the student if one does not define it carefully, which is the aim of this short note.

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First let recall that the supersymmetry algebra contains

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2\sigma^\mu{}_{\alpha\dot\alpha} P_\mu \tag{1}$$

using Weyl notation, i.e. $\bar{Q}_{\dot\alpha} = (Q_\alpha)^* \equiv Q^*_\alpha$ (when using quantum operator this becomes $\bar{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$). Multiplying this relation on the left by $\bar\sigma^\mu$ and using

$$\sigma^\mu{}_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha} = \mathrm{tr}(\sigma^\mu\bar\sigma^\nu) = -2\eta^{\mu\nu} \tag{2}$$

gives

$$P_\mu = -\frac{1}{4}\bar\sigma^{\mu\dot\alpha\alpha}\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\}. \tag{3}$$

So we can computes the energy $H = P^0 = -P_0$:

$$H = \frac{1}{4}\sum_{\alpha=1,2}\{Q_\alpha, Q^*_\alpha\} \tag{4}$$

since $\bar\sigma^0 = \mathrm{id}$. Then we evaluate the expectation value of the energy on the vacuum $|0\rangle$:

$$\langle 0| H |0\rangle = \sum_{\alpha=1,2}\langle 0| \{Q_\alpha, Q^*_\alpha\} |0\rangle = \sum_{\alpha=1,2}\left(|Q^*_\alpha |0\rangle|^2 + |Q_\alpha |0\rangle|^2\right) \geq 0 \tag{5}$$

since the Hilbert space comes with a positive scalar product. We arrive at the conclusion that this expectation value should be positive since the right hand side is a sum of squares. Now every book or review conclude that the energy should be positive in supersymmetric theories, and this conclusion could seem confusing: after all, we have already learn that in non-gravitational theories the energy origin is arbitrary and can be shifted by an arbitrary constant. But before coming at the resolution of this problem, let take a detour and bring here some other results on this "energy" positivity.

The scalar potential is also given by the sum of squared terms:

$$V(\phi_i) = \sum_{i,a}(|F_i|^2 + |D_a|^2) \geq 0 \tag{6}$$

where the F_i and D_a are the F- and D-terms from the chiral (labelled by i) and vector (labelled by a) multiplets (they depends only on the scalar fields ϕ_i). Again one sees that this potential is positive and assimilates it to the total energy, so this leads to the same conclusion. But here is a clue of what can be wrong by speaking about energy, since we are speaking of different kind of energies, so care is needed to define them.

Now write the supersymmetric lagrangian for a single massless chiral multiplet

$$L_0 = -|\partial\phi|^2 - \frac{1}{2}\bar\psi\partial\!\!\!/\psi \tag{7}$$

switching to Dirac notation (ψ is a Majorana spinor, ϕ a complex scalar field), and define the associated energy–momentum tensor

$$T_{0,\mu\nu} = \sum_A \frac{\delta L_0}{\delta(\partial^\mu\Phi_A)}\partial_\nu\Phi_A + \eta_{\mu\nu}L_0 \tag{8}$$

computed thanks to the Noether theorem, where $\Phi_A = \{\phi, \phi^\dagger, \psi, \bar{\psi}\}$. Here the energy is the conserved charged defined as the integral over the space of the $(0,0)$ component:

$$H_0 = P^0 = \int \mathrm{d}^3 x \, T^0_{\ 0} \, . \tag{9}$$

In principle this H_0 is the generator of time translation and should be assimilated to the one which appears in the the supersymmetry algebra (1).

In the usual scalar field theory, one has problem when trying to compute the energy because of the zero modes which contribute as an infinite amount; one then adds an infinite constant $-\Omega_0$ and finetune it to remove this infinity, and then forget about it (imposing normal order and so on). But in the case of a supersymmetric lagrangian, the zero-point energies from fermions and bosons cancel; this can be seen by considering only one mode of each type. The corresponding hamiltonian is the usual one from harmonic oscillators (a is for the bosonic mode, b for the fermionic):

$$H = \frac{\hbar\omega}{2} \left(\left\{ a^\dagger, a \right\} + \left[b^\dagger, b \right] \right) . \tag{10}$$

Using the commutation for a and anticommutation for b this gives

$$H = \frac{\hbar\omega}{2} \left(2a^\dagger a + 1 + 2b^\dagger b - 1 \right)$$

and then

$$H = N_b + N_f, \qquad N_b = a^\dagger a, N_f = b^\dagger b. \tag{11}$$

This extends easily to field theory with an infinite number of modes. It is not anymore useful to introduce a constant. But nothing prevent us to do so [1], so let subtract an arbitrary constant Ω_0 to this lagrangian and write

$$L = L_0 - \Omega_0 \tag{12}$$

The energy–momentum tensor is then modified by a constant

$$T_{\mu\nu} = T_{0,\mu\nu} + \Omega_0. \tag{13}$$

Integrating over the space gives the modified energy (putting the system in a box of volume V to regulate the integral)

$$H = H_0 + \Omega_0 V \Longrightarrow \langle 0| H |0\rangle = \langle 0| H_0 |0\rangle + \Omega_0 V \tag{14}$$

So which generator should we use? H or H_0? The answer is of course H_0. In fact, this energy arises only from the particles (i.e. the fields) present in the system, and the scalar potential (6) has the same origin. And this decomposition of the generators in several parts is always possible (interactions mixes only particle parts). So it is meanningful to speak about "positivity" only for the scalar potential, which is also the correct order parameter for supersymmetry breaking; the total energy is arbitrary.

[1]Even in the superspace formalism where $L = K(\Phi, \Phi^\dagger)|_D + W(\Phi)|_F + \text{h.c.} - \Omega_0$.