Lagrangian mechanics and special relativity

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Sommaire

Introduction

Organization

The lecture is composed ot two parts.

Introduction to Lagrangians in the context of non-relativistic systems

In this chapter, we have $v \ll c$ where v is the velocity of the observer and $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$.

Non-relativistic systems are described by classical mechanics and Newtons equations of motion.

All systems

He re we have $v \leq c$.
Examples : photons, cosmics rays...

The dynamic of relativistic systems is very different. All intuitions fails. It is the domain of all modern physics :

- cosmology ;
- string theory ;
- physics of the early universe ;
- quantum field theory.

Lagrangian theory also applies for relativistic systems, but we will develop it mainly for non-relativistic systems in the first part.

Why revisit classical mechanics?

Becaus we will introduce very powerfull, different method to determine the dynamics of a particle.

Part I

Introduction to Lagrangians in the context of non-relativistic systems

Chapter 1

Introduction: classical mechanics with Newtons equations against Lagrangian approach

1.1 Limitations of Newtons approach

Newtonians classical mechanics has (at least) three limitations :

1. It describes particles, which is inconvenient because often we want to study the dynamics of a number of particles, or the dynamics of extended objects (a spinning top, a bicycle. . .), or the dynamics of fields.
2. Newtons equations are formulated in a special coordinate system — cartesian coordinates in an inertial frame.
   Let x_i $(i = 1, 2, 3)$ be the coordinates of a particle of mass m in a inertial frame. Then, under the influence of a force with components $f_i(t)$, the dynamics of the particle is obtained by solving Newtons equations
   $$m\ddot{x}_i(t) = f_i(t) \tag{1.1}$$
3. You need to find the force f_i in order to solve Newtons equations: this is not always easy (think to the double pendulum — figure ??)!

In principle, one can overcome these limitations:

1. For N particles you could write down N coupled newton's equations and try to solve them.
2. One can pass from cartesian coordinates x_i to other coordinate systems by using the "chair rule" of differentiation.
3. In principe it's possible to find the f_i, but it's difficult.

Conclusion: got rid of Newtons equations and approach classical mechanics differently: Lagrangian method which does not have the limitations.

Lagrangian approach to classical mechanics is valid for any number of particles in any coordinate system, and you don't need to know the f_i.

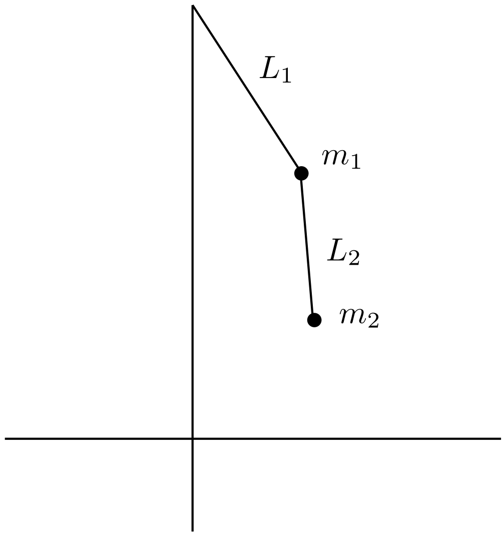

Figure 1.1: Double pendulum.

1.2 Difference between Newtonian classical mechanics and Lagrangian classical mechanics

– Newtons equations are vector equations. Lagrangian approach puts great emphasis on scalar quantities.
  Equation for a particle with m and the position $\boldsymbol{x}(t)$:
  – $\ddot{\boldsymbol{x}}(t)$ and $\boldsymbol{f}$ are vectors (equation (1.1)) ;
  – kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{x}}^2$ and potential energy V are scalar.
– In Newtonian mechanics, to determine the dynamics of a particle, one has to solve the second order equation (1.1). You need to specify $\boldsymbol{x}(t_i)$ and $\dot{\vec{x}}(t_i)$.
  In Lagrangian mechanics the view point is totally different:
  – We said that cartesian coordinates are not always the most useful : get rid of them! Introduce the *configuration space*, which consist of the generalised coordinates q_i of a system.
    Exemple 1.1.
    For one point (cartesian coordinates)

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$

    Exemple 1.2.
    For two points (spherical coordinates):

$$\begin{aligned} q_1 = r, \quad q_2 = \theta, \quad q_3 = \varphi \\ q_4 = R, \quad q_5 = \alpha, \quad q_6 = \beta \end{aligned}$$

  – We said that in Newtonian mechanics $\boldsymbol{x}(t_i)$, $\dot{\boldsymbol{x}}(t_i)$ and $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$.
    In Lagrangian mechanics the approach is very different. At time t and $t+\delta t$, the generalised coordinates take value $q_\alpha(t)$ and $q_\alpha(t+\delta t)$. As the

system we are describing evolves in time, it describes a trajectory/curve in configuration space.

The trajectory in configuration space is defined uniquely by specifying either $q_\alpha(t_i)$ and $\dot{q}_\alpha(t_i)$ (Newtonian approach), or $q_\alpha(t_i)$ and $q_\alpha(t_f)$ (Lagrangian approach).

The path followed is the one which minimizes a quantity (a functional) $S[q_\alpha]$. $S[q_\alpha]$ is called the *action* and is directly related to the Lagrangian.

Remarque : $S[q_\alpha]$ is a functional of q_α, to be distinguished from functions $f(q_\alpha)$.

The following questions are:

– What is a functional?
– How does one find the trajectory which minimize a functional? (the trajectory must satisfy Euler–Lagrange equations)
– What is the definition of the action $S[q_\alpha]$? (We will see that it's related to the Lagrangian, which itself depends on the scalar T and V)
– How two shows that we have the same solutions as that we would get from Newton's equations?

Chapter 2

Calculus of variations: The Lagrangian, Euler-Lagrange equations

2.1 Functionals and the Euler–Lagrange equations

2.1.1 Examples

1. Consider 2-dimensional Euclidean space coordinates (x, y). Set two points A and B. The distance between A and B is an example of a functional:
   – functional: $L[y]$;
   – path : $y(x)$.
   The path which minimizes $L[y]$ is a straight line.
2. Two points A and B on the surface of the sphere. Functional $L(\theta]$ with path $\theta(\varphi)$. A geodesic minimizes $L[\theta]$.
3. The brachistochrone.

   Consider two points A and B in the (x, z) plane, and a wire of shape $z(x)$ linking A and B.

   Place bead of mass m at A, and under the influence of gravity, it goes to B (neglect all frictional forces).

   Which path $z(x)$ minimizes the time it takes for the bead to go from A to B?
   – functional: $T[z]$;
   – path : $z(x)$.
4. A place takes off from New-York bound for Paris. The on-board computer must choose the optimal path $\boldsymbol{x}(t)$ (sequence of latitudes, longitudes and altitudes) such that, given the wind direction, the total of fuel used is minimum.
   – functional: $V_F[\boldsymbol{x}]$;
   – path : $\boldsymbol{x}(t)$.

In all of the examples, one has to minimize $F[q_\alpha]$ which depends on a path $q_\alpha(t_i)$.

For the different examples:

1. $F = L$, $\alpha = 1$
2. $F = L$, $\alpha = 1$
3. $F = T$, $\alpha = 1$
4. $F = V_F$, $\alpha = 3$

From the examples, you can see that the functional $F[q_\alpha]$, the path $q_\alpha(\tau)$ and the parameter τ, all change from one example to the next.

We will introduce the calculus of variations for a functional $F[\boldsymbol{x}]$ with path $\boldsymbol{x}$.

It's very easy to translate the results we will obtain from:

– $\boldsymbol{x}(t) \to q_\alpha(\tau)$
– $\boldsymbol{F}(x) \to F[q_\alpha]$

2.2 Introduction to functional

Given a path $\boldsymbol{x}(t)$, the simplest functionals are integrals along the path of $\boldsymbol{x}(t)$ and its derivatives.

Exemple 2.1.

–
$$F_1[\boldsymbol{x}(t)] = \int_{t_i}^{t_f} \mathrm{d}t(\boldsymbol{x}^2)$$

–
$$F_2[\boldsymbol{x}(t)] = \int_{t_i}^{t_f} \mathrm{d}t(\boldsymbol{x} \cdot \dot{\boldsymbol{x}})$$

Attention : A functional is a scalar.

$f(\cdot)$	$F[\cdot]$
$f : \mathbb{R} \to \mathbb{R}$	$f : \mathcal{F} \to \mathbb{R}$
$x \mapsto f(x)$	$f \mapsto F[f]$
$\delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta x_i$	$\delta F = \int \frac{\delta F[f]}{\delta f(x)} \delta f(x)\, \mathrm{d}x$

Table 2.1: Differences between a function and a functional

The extremum for a functional is given by

$$\frac{\delta F}{\delta f} = 0 \tag{2.1}$$

2.2.1 Examples of functional derivatives

1. Consider

$$F[f] = \int f(x)\mathrm{d}x$$

so

$$\delta F = \int \delta f(x) \mathrm{d}x$$

2. Consider

$$F[f] = a + \int b(x) f(x) \mathrm{d}x + \frac{1}{2} \int c(x,y) f(x) f(y) \mathrm{d}x \mathrm{d}y$$

so

$$\begin{aligned}
\delta F &= \int b(x) \delta f(x) \mathrm{d}x + \frac{1}{2} \int c(x,y) [\delta f(x) f(y) + \delta f(y) f(x)] \mathrm{d}x \mathrm{d}y \\
&= \int b(x) \delta f(x) \mathrm{d}x + \frac{1}{2} \int c(x,y) [f(y) + f(y)] \delta f(x) \mathrm{d}x \mathrm{d}y \\
&= \int b(x) \delta f(x) \mathrm{d}x + \frac{1}{2} \int \delta f(x) \left[\int c(x,y) f(y) \mathrm{d}y \right] \mathrm{d}x
\end{aligned}$$

and

$$\delta F = b(x) + \int c(x,y) f(y) \mathrm{d}y$$

2.3 Euler–Lagrange equations

We will derive the Euler–Lagrange equations for a function $F[\boldsymbol{x}]$ with path $\boldsymbol{x}(t)$. We define

$$F_1[\boldsymbol{x}] = \int |\boldsymbol{x}|^2 \mathrm{d}t \tag{2.2}$$

$$F_3[\boldsymbol{x}] = \int \boldsymbol{x}\dot{\boldsymbol{x}} \mathrm{d}t \tag{2.3}$$

$$F_3[\boldsymbol{x}] = \int |\dot{\boldsymbol{x}}|^2 \mathrm{d}t \tag{2.4}$$

A general functional takes the form

$$F[\boldsymbol{x}] = \int_{t_i}^{t_f} f(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) \mathrm{d}t \tag{2.5}$$

Remarques :

1. We suppose that $f(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$ does not depend on higher time derivatives.
2. We will use the summation convention.

How does we find the path $\hat{\boldsymbol{x}}(t)$ which minimizes $F[\boldsymbol{x}]$ subject to the conditions

$$\boldsymbol{x}(t_i) = \boldsymbol{x_i} \qquad \boldsymbol{x}(t_f) = \boldsymbol{x_f}$$

$\hat{\boldsymbol{x}}$ is the path we are looking for. Consider a small variation $\boldsymbol{\eta}(t)$ about that path

$$\boldsymbol{x}(t) = \hat{\boldsymbol{x}}(t) + \boldsymbol{\eta}(t) \tag{2.6}$$

where we suppose that

$$\boldsymbol{\eta}(t_i) = 0 \qquad \boldsymbol{\eta}(t_f) = 0$$

then

$$
\begin{aligned}
F[\boldsymbol{x}] &= \int_{t_i}^{t_f} f\Big(\hat{\boldsymbol{x}}(t)+\boldsymbol{\eta}(t), \dot{\hat{\boldsymbol{x}}}(t)+\dot{\boldsymbol{\eta}}(t), t\Big)\mathrm{d}t \\
&= \int_{t_i}^{t_f} \left[f\Big(\hat{\boldsymbol{x}}(t), \dot{\hat{\boldsymbol{x}}}(t), t\Big) + \frac{\partial f}{\partial \boldsymbol{x}}\boldsymbol{\eta} + \frac{\partial f}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{\eta}}\right]\mathrm{d}t \\
&= F[\hat{\boldsymbol{x}}] + \int_{t_i}^{t_f} \left(\frac{\partial f}{\partial \boldsymbol{x}}\boldsymbol{\eta} + \frac{\partial f}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{\eta}}\right)\mathrm{d}t \\
&= F[\hat{\boldsymbol{x}}] + \int_{t_i}^{t_f} \frac{\partial f}{\partial \boldsymbol{x}}\boldsymbol{\eta}\mathrm{d}t - \int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\right)\boldsymbol{\eta}\mathrm{d}t
\end{aligned}
$$

because

$$
\begin{aligned}
\int_{t_i}^{t_f} \frac{\partial f}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{\eta}}\mathrm{d}t &= \underbrace{\left[\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\boldsymbol{\eta}\right]_{t_i}^{t_f}}_{=0} - \int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\right)\boldsymbol{\eta}\mathrm{d}t \\
&= -\int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\right)\boldsymbol{\eta}\mathrm{d}t
\end{aligned}
$$

We find

$$
\delta F = \int_{t_i}^{t_f} \left[\frac{\partial f}{\partial \boldsymbol{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\right)\right]\boldsymbol{\eta}\,\mathrm{d}t \tag{2.7}
$$

where

$$
\boldsymbol{\eta}(t) = \delta\boldsymbol{x}(t)
$$

The functional has an extremum if $\delta F = 0$ for all infinitesimal $\boldsymbol{\eta} = \boldsymbol{x}$. Assuming η_i are all independent, it follows that the Euler–Lagrange equation is

$$
\boxed{\frac{\partial f}{\partial \boldsymbol{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\boldsymbol{x}}}\right) = 0} \tag{2.8}
$$

Remarques :

1. The Euler–Lagrange equations are invariant under
   - $f \mapsto f' = f + c$
   - $f \mapsto f' = f + \dfrac{\mathrm{d}\Lambda(q_\alpha, \tau)}{\mathrm{d}\tau}$
2. Euler–Lagrange equations hold in any coordinate system q_α (for example, the q_α could be a non interual coordinate system).

Exemple 2.2 (Shortest distance on a plane)**.**
Which path $y(x)$ is the shortest (in distance) between A and B?

$$
\begin{aligned}
L[y] = \int \mathrm{d}\ell &= \int_A^B \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \\
&= \int_A^B \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\
&= \int_{x_a}^{x_b} f(x, y, \mathrm{d}y/\mathrm{d}x)\mathrm{d}x
\end{aligned}
$$

The Euler–Lagrange equation is

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}$$

where

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad f = \sqrt{1+y'^2}$$

and

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\\ &\Longrightarrow \left(\frac{y'}{\sqrt{1+y'^2}}\right) = c\\ &\Longrightarrow y' = c\\ &\Longrightarrow y = cx+k\end{aligned}$$

We need to fix constants such that the straight line passes through A and B.

Exemple 2.3 (Brachistochrone)**.**

Which curve $y(x)$ minimizes the total time taken for the particles to go from $A(0,0)$ to $B(x_0, -y_0)$ ($y_0 > 0$) under the effect of gravity?

We have

$$\begin{aligned}T_{A\to B} &= \int_A^B \frac{\mathrm{d}\ell}{v(\ell)}\\ &= \int_A^B \frac{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}}{\sqrt{2gy}} \qquad \text{Energy conservation}\\ &\underbrace{=}_{x'=\frac{\mathrm{d}x}{\mathrm{d}y}} \int_0^{-y_0} \frac{\sqrt{1+x'^2}}{\sqrt{2gy}}\,\mathrm{d}y\\ &= \int_0^{-y_0} f(x,x',y)\mathrm{d}y\end{aligned}$$

since f does not depend explicitly on x,

$$\frac{\partial f}{\partial x} = 0 \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial f}{\partial x'}\right) = 0 \Longrightarrow \frac{\partial f}{\partial x'} = cste$$

where

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{1+x'^2}}\frac{1}{\sqrt{2gy}} = c$$

Chapter 3

Calculus of variations: symmetry and conservation equations

3.1 Conservation

In nearly all physical conservation phenomena, there exists quantities which are conserved during the evolution of the system.

- How do these conserved quantities appear in the functional approach?
- We will work with a function $F[q_\alpha] = \displaystyle\int_{t_i}^{t_f} f(q_\alpha, \dot{q}_\alpha, t)$.

Définition 3.1. A function $C(q_\alpha(t), \dot{q}_\alpha(t), t)$ is constant in time [1] if, along the path $\hat{q}_\alpha(t)$ which is solution yo the Euler–Lagrange equations its total time derivative vanishes

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 0 \tag{3.1}$$

3.1.1 Time independance

Suppose that f does not depend explicitly on t.

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha + \frac{\partial f}{\dot{q}_\alpha}\ddot{q}_\alpha \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{q}_\alpha}\dot{q}_\alpha\right)\end{aligned}$$

So we obtain

$$h = \frac{\partial f}{\partial \dot{q}_\alpha}\dot{q}_\alpha - f \tag{3.2}$$

is conserved if $\partial f/\partial t = 0$ [2].

1. It is the same thing as bo be a constant of motion, or a conserved quantity.
2. In french, the name is "identité de Beltrami".

Exemple 3.1.
Minimal distance between two points in the plane.

$$\begin{aligned} L &= \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \\ &= \sqrt{1 + y'^2}\,\mathrm{d}y \end{aligned}$$

Here $\partial f / \partial x = 0$, so

$$\begin{aligned} h &= \frac{\partial f}{\partial y'} y' - f = c \\ &= \frac{y'}{\sqrt{1+y'^2}} y' - \sqrt{1+y'^2} \\ &= \frac{1}{\sqrt{1+y'^2}} (y'^2 - (1 + y'^2)) \\ &\Longrightarrow y'^2 = cste \end{aligned}$$

3.1.2 Coordinate independance

Suppose that f does not depend explicitly on q_α

$$\frac{\partial f}{\partial q_\alpha} = 0$$

then, from the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{q}_\alpha} = \frac{\partial f}{\partial q_\alpha} = 0$$

Thus $\partial f / \partial \dot{q}_\alpha$ is a conserved quantity.

3.2 Noether's theorem

Consider a 1-parameter family of transformations

$$q_\alpha(t) \longmapsto Q_\alpha(s, t) \qquad s \in \mathbb{R}$$

where $Q_\alpha(0, t) = q_\alpha(t)$.

Exemple 3.2.
q_α are cartesian coordonates

$$\boldsymbol{x} \longmapsto \boldsymbol{x} + s\boldsymbol{n}$$

This transformation is said to be a continuous symmetry of f if it leaves f invariant, or more explicitly

$$\frac{\partial f}{\partial s}(Q_\alpha(s, t), t)\bigg|_{s=0} = 0$$

Théorème 3.1 (Noether's theorem)**.** For every continuous symmetry there exists a conserved quantity.

Chapter 4

The Lagrangian of classical mechanics

Recall: the Lagrangian formulation of classical mechanics introduces:
- configuration space with generalized coordinates q_α (supposed independent);
- the path which the system follows in time as it evolves dynamically is that which minimizes the functional $S[q_\alpha]$ called the action;
- hence the path followed is that which satisfies the Euler–Lagrange equations (2.8).

Question: Which Lagrangian gives us back Newtons laws?
Answer: There is no general rule to write down L, valid for relativistic, non-relativistic, etc., systems.

But for the majority of non-relativistic systems, a Lagrangian of the form

$$\boxed{L = T - V} \tag{4.1}$$

where T is the kinetic energy and V the potential energy, gives the correct equations of motion.

Exemple 4.1.

Let's check that this Lagrangian "works" in the simplest case: a particle of mass m, moving in a conservative potential V, and which we describe using cartesian coordinates in an inertial frame.

For this particle

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x})$$

then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) &= \frac{\partial L}{\partial \boldsymbol{x}} \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\boldsymbol{x}}) &= -\nabla V \\ m\ddot{\boldsymbol{x}} &= -\nabla V \end{aligned}$$

Remarques :

1. The Lagrangian depends on physical (scalar) quantities T and V, which are completely independent of the choice of the generalized coordinates.
2. The Lagrangian is not defined uniquely

$$L' = L + const$$
$$L' = L + \frac{\mathrm{d}\Lambda(q_\alpha, t)}{\mathrm{d}t}$$

give the same equations of motions.
3. Conserved quantities: if L is explicitly time independent, then

$$h = \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$$

is constant. h corresponds to the total energy: $h = T + V$, if
   (a) V is independent of $\dot{q}_\alpha$;
   (b) T is quadratic in $\dot{q}_\alpha$, in other words

$$T = \frac{1}{2}m\, g_{\alpha\beta}(q_\alpha)\; \dot{q}_\alpha \dot{q}_\beta$$

Exemple 4.2.

i. Particle of mass m in cartesian coordinates

$$T = \frac{1}{2}m\dot{\boldsymbol{x}}^2 = \frac{1}{2}m(\dot{\boldsymbol{x}}_{\boldsymbol{1}}{}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{2}}{}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{3}}{}^2)$$

with $g_{ij} = \delta_{ij}$.

ii. Particle of mass m in spherical polar coordinates (r, θ, φ)

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta)$$

with $g_{11} = 1$, $g_{22} = r^2$ and $g_{33} = r^2 \sin^2\theta$.

4.1 Exemples

4.1.1 Ball slides

A ball slides (without friction) on a wire of shape $y = Ax^2$ $(A > 0)$ under the effect of gravity.

1. Write down the Lagrangian.
2. Calculate the Euler–Lagrange equation.
3. Are there any conserved quantities?

Lets choose x as generalized coordinate.

$$L = T - V = h(x, \dot{x}, t)$$

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad \text{where } y = Ax^2 \\
&= \frac{1}{2}m(\dot{x}^2 + 4A^2x^2\dot{x}^2) \\
&= \frac{1}{2}\dot{x}^2(1 + 4A^2x^2)
\end{aligned}
$$

$$
V = mgy = mgAx^2
$$

then

$$
L = \frac{1}{2}m\dot{x}^2(1 + 4A^2x^2) - mgAx^2
$$

Conserved quantity: $\partial_t L = 0$ and T is quadratic in $\dot{x}$, V is independent of $\dot{x}$, so the total energy is conserved (by Beltrami)

$$
h = \frac{1}{2}m\dot{x}^2(1 + 4A^2x^2) + mgAx^2
$$

Equations of motion: let's calculate the Euler–Lagrange equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x} \\
\ddot{x}(1 + 4A^2x^2) &= -4\dot{x}A^2x - 2gAx
\end{aligned}
$$

You can obtain the same equation with $\mathrm{d}h/\mathrm{d}t = 0$.

In order to solve for $x(t)$, it is much simple to use the conserved quantity h: in that case, you "only" need to solve a first order differential equation.

4.1.2 Simple pendulum with variable height

Simple pendulum in which the point of support of the pendulum changes height with time.

1. Write down Lagrangian.
2. Determine equations of motion and any conserved quantity.

Let's choose θ as generalized coordinate.

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\dot{y}^2 \\
&= \frac{1}{2}m\ell^2(\cos\theta)^2\dot{\theta}^2 + \frac{1}{2}m\Big(\dot{h}^2 + 2\dot{h}\ell(\sin\theta)\dot{\theta} + 2h\ell(\sin\theta)^2\dot{\theta}^2\Big) \\
&= \frac{1}{2}m\ell\dot{\theta}^2 + \frac{1}{2}m\Big(\dot{h}^2 + 2\dot{h}\ell(\sin\theta)\dot{\theta}\Big)
\end{aligned}
$$

and

$$
V = mgy = mg\Big(h(t) - \ell\cos\theta\Big)
$$

then

$$
L = \frac{1}{2}m\Big(\ell^2\dot{\theta}^2 + \dot{h}^2 + 2\dot{h}\ell(\sin\theta)\dot{\theta}\Big) - mg\Big(h(t) - \ell\cos\theta\Big)
$$

Equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\ell^2\dot{\theta} + m\dot{h}\ell\sin\theta) = m\dot{h}\ell\dot{\theta}\cos\theta - mg\ell\sin\theta$$

$$\ell^2\ddot{\theta} + \ddot{h}\ell\sin\theta + \dot{h}\ell\dot{\theta}\cos\theta = \dot{h}\ell\dot{\theta}\cos\theta - g\ell\sin\theta$$

$$m\ell\ddot{\theta} = -m(g+\ddot{h})\sin\theta$$

It's equivalent to a standard simple pendulum in a gravitational field $g + \ddot{h}$.

4.1.3 Free particle in non-inertial frame

Free particle of mass m in three spatial dimensions. Its lagrangian in an inertial frame

$$L = \frac{1}{2}m(x^2 + y^2 + z^2)$$
$$= \frac{1}{2}m\dot{\boldsymbol{r}}^2$$

What is the dynamics of the particle in a coordinate system which is rotating relative to this one? It will be subjected to so-called "fictitious forces" which are forces which result from the acceleration of the coordinate system itself (rather than any physical force which acts on the particle).

Lagrangian formulation provides a particulary straightforward way of determining equations of motion in the news coordinate system (one simply has to write down the Lagrangian in new coordinate system and then write down the Euler–Lagrange equations).

Let $\boldsymbol{r}' = (x', y', z')$ be the new coordinates, which share the same origin as the old coordinates, but they rotate with angular velocity

$$\boldsymbol{\omega} = (0, 0, \dot{\theta}(t))$$

where θ is a given function of t.

$$z' = z$$
$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

We need to invert to find (x, y, z) in terms of (x', y', z')

$$z' = z$$
$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

We need $\dot{x}$, $\dot{y}$ and $\dot{z}$.

$$\dot{z} = \dot{z}'$$
$$\dot{x} = \cos\theta(\dot{x}' - y'\dot{\theta}) - \sin\theta(\dot{y}' + x'\dot{\theta})$$
$$\dot{y} = \cos\theta(\dot{y}' + x'\dot{\theta}) - \sin\theta(\dot{x}' - y'\dot{\theta})$$

substituting into

$$\begin{aligned} L &= \frac{1}{2}m(x^2+y^2+z^2) \\ &= \frac{1}{2}m\Big(\dot{z}' + (\dot{x}' - y'\dot{\theta})^2 + (\dot{y}' + x'\dot{\theta})^2)\Big) \\ &= \frac{1}{2}m\Big(x'^2 + y'^2 + z'^2 + 2\dot{\theta}(x'\dot{y}' - y'\dot{x}') + \dot{\theta}^2(x'^2+y'^2)\Big) \end{aligned}$$

The Euler–Lagrange equations in the (x', y', z') coordinate system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{r}}'} = \frac{\partial L}{\partial \boldsymbol{r}'}$$

In terms of $\boldsymbol{r}$

$$L = \frac{1}{2}m\Big(\dot{\boldsymbol{r}}'^2 + 2\boldsymbol{\omega}\cdot(\boldsymbol{r}'\times\dot{\boldsymbol{r}}') + (\boldsymbol{\omega}\times\boldsymbol{r}')^2\Big)$$

then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(2\dot{\boldsymbol{r}}' + 2(\boldsymbol{\omega}\times\boldsymbol{r}')\Big) &= 2(\dot{\boldsymbol{r}}\times\boldsymbol{\omega}) - 2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') \\ \ddot{\boldsymbol{r}}' &= \underbrace{-\dot{\boldsymbol{\omega}}\times\boldsymbol{r}'}_{\text{Euler}} - \underbrace{2(\boldsymbol{\omega}\times\dot{\boldsymbol{r}}')}_{\text{Coriolis}} - \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')}_{\text{Centrifugal}} \end{aligned}$$

4.1.4 Spherical pendulum

Spherical pendulum. A pendulum of mass m on a rod of length ℓ of negligible mass. The pendulum can move in 3D.

Choose as generalized coordinates $\theta \in [0, \pi[$ and $\varphi \in [0, 2\pi[$.

$$T = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2\sin^2\theta)$$

$$V = -mg\ell\cos\theta$$

$$L = T - V = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2\sin^2\theta) + mg\ell\cos\theta$$

Conserved quantities: $\partial_t L = 0$ and T is quadratic in φ and θ, hence the total energy is

$$H = T + V = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2\dot{\varphi}^2\sin^2\theta) - mg\ell\cos\theta$$

is conserved.

Equations of motion

– For φ

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}} &= \frac{\partial L}{\partial\varphi} = 0 \\ \implies \frac{\partial L}{\partial\dot{\varphi}} &= cste \\ &= \dot{\varphi}m\ell\sin^2\theta = J_z \end{aligned}$$

– For θ

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} &= \frac{\partial L}{\partial\theta} \\ \implies m\ddot{\theta} &= \frac{-\partial V_{eff}}{\partial\theta} \end{aligned}$$

4.2 Consequences of Noether's theorem

Recall: for each transformation $q_\alpha(t) \to Q_\alpha(s,t)$ which leaves L invariant to linear order s (i.e continuous symmetry of L), there is a conserved quantity

$$\frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial Q_\alpha}{\partial s}\bigg|_s = cste$$

Consider N particles interaction between each other via a potential V

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\boldsymbol{r}}_i^2 - V \tag{4.2}$$

where V only depends on the relative coordinates of the particles $V(\{\boldsymbol{r}_i - \boldsymbol{r}_j\})$

4.2.1 Momemtum conservation

Proposition 4.1. Homogeneity of space: the system is invariant under spatial translations.

The Lagrangian (4.2) has translational symmetry

$$L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i) = L(\boldsymbol{r}_i + s\boldsymbol{n}, \dot{\boldsymbol{r}}_i)$$

From Noether's theorem, the conserved quantity is

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} \cdot \boldsymbol{n} = cste$$
$$\iff \boldsymbol{P}_{tot} \cdot \boldsymbol{n} = cste \tag{4.3}$$

If system is invariant for any $\boldsymbol{n}$, then the total momentum $\boldsymbol{P}_{tot}$ is conserved. If system is invariant only for a specific $\boldsymbol{n}$, for example $\boldsymbol{n} = \boldsymbol{e}_z$, then the $\boldsymbol{P}_{tot}|_z$ is conserved.

4.2.2 Angular momemtum conservation

Proposition 4.2. Isotropy of space: the system is invariant under spatial rotations.

The Lagrangian (4.2) is invariant under rotations about any axis $\boldsymbol{n}$. All the $\boldsymbol{x}_i$ are rotated by the same angle $\delta\theta = s$, then

$$\boldsymbol{x}_i \longrightarrow \boldsymbol{x}'_i = \boldsymbol{x}_i + \delta\theta(\boldsymbol{n} \times \boldsymbol{x}_i) \tag{4.4}$$

so

$$L(\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t) = L(\boldsymbol{x}_i + \delta\theta(\boldsymbol{n} \times \boldsymbol{x}_i), \dot{\boldsymbol{x}}_i + \delta\theta(\boldsymbol{n} \times \dot{\boldsymbol{x}}_i), t) \tag{4.5}$$

The corresponding conserved quantity is

$$cste = \sum_i \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i} \frac{\partial \boldsymbol{x}'_i}{\partial \theta}\bigg|_{\theta=0}$$
$$= \sum_i m_i \dot{\boldsymbol{x}}_i (\boldsymbol{n} \times \boldsymbol{x}_i)$$
$$cste = \boldsymbol{nJ} \tag{4.6}$$

and

$$\boldsymbol{J} = \sum_i \dot{\boldsymbol{x}}_i \times m_i \boldsymbol{x}_i = \sum_i \boldsymbol{x}_i \times \dot{\boldsymbol{p}}_i \tag{4.7}$$

If system is invariant under rotations about any vector $\boldsymbol{n}$, then the total angular momentum $\boldsymbol{J}_{tot}$ is conserved.

Chapter 5

Smalls oscillations and normal modes

> "Physics is that subject of human experience which can be reduced to coupled harmonic oscillations."
> — Peskin

- Here we will dealed with uncoupled harmonic oscillators.
- The Lagrangian formulation of classical mechanics gives us a relatively straightforward way in which to write down the dynamical equations of motion for a system. But, actually solving those equations is generally very difficult: one has to use some kind of simplication or approximation schemes.
- Here we study (exactly) the dynamics of a system only about its equilibrium positions.
- If the system is disturbed slightly about a stable equilibrium position, it will oscillate with small amplitude oscillations.
- In the Lagrangian formulation, we can study the dynamics of these oscillations and the problem reduced to studying N uncoupled harmonic oscillators, each ringing with a different frequency ω_α.
- These uncoupled oscillating modes are called *normal modes*.

5.1 The equations

We work with generalized coordinates q_α. Potential is $V = V(q_\alpha)$ (so V is explicitly independent of t, q_α). Kinetic energy is of the form $T = 1/2\ m_{\alpha\beta}(q_\alpha)\dot{q}_\alpha\dot{q}_\beta$. From the Euler–Lagrange equations, the system is in equilibrium when

$$\left.\frac{\partial V}{\partial q_\alpha}\right|_{q_{\alpha,0}} = 0 \tag{5.1}$$

If initially the generalized coordinates take values $q_{\alpha,0}$ and if $\dot{q}_\alpha = 0$, then the system stays in equilibrium indefinitely.

What happens when the system is displaced from an equilibrium position?

- Stable equilibrium: small displacement → small bounded oscillations ($e^{i\omega t}$, with $\omega \in \mathbb{R}$).

– Unstable equilibrium: small displacement → unbounded oscillations ($e^{\omega t}$, with $\omega \in \mathbb{R}_+$).
– Partially equilibrium:
All will be classified according to the signs of

$$\frac{\partial^2 V}{\partial q_\alpha q_\beta} \tag{5.2}$$

Aim: departures from equilibrium are small: Taylor expand the Lagrangian about an equilibrium position, heeping only lowest order terms

$$q_\alpha(t) = q_{\alpha,0} + \eta_\alpha(t) \tag{5.3}$$

where $q_\alpha \ll 1$, and $q_{\alpha,0}$ is the equilibrium position, and is solution of (5.1).

$$\begin{aligned} V(q_\alpha) &= V(q_{\alpha,0} + \eta_\alpha) \\ &= V(q_{\alpha,0}) + \underbrace{\left.\frac{\partial V}{\partial q\alpha}\right|_{q_{\alpha,0}} \eta_\alpha}_{=0} + \frac{1}{2} \left.\frac{\partial^2 V}{\partial q_\alpha q_\beta}\right|_{q_{\alpha,0}} \eta_\alpha \eta_\beta \end{aligned}$$

Set $V(q_{\alpha,0})$ to zero by choosing the zero of potential energy at the equilibrium position.

Rewritten in matrix form

$$V = \frac{1}{2} \underline{\eta}^t \underline{\underline{V}}\, \underline{\eta} \tag{5.4}$$

where $\underline{\underline{V}}$ is a matrix with components $V_{\alpha\beta}$ (5.2), and

$$\underline{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

Kinetic energy is

$$T = \frac{1}{2} m_{\alpha\beta}(q_\alpha) \dot{q}_\alpha \dot{q}_\beta$$

where $\dot{q}_\alpha(t) = \dot{q}_{\alpha,0} + \dot{\eta}_\alpha(t) = \dot{\eta}_\alpha(t)$. So

$$\begin{aligned} T &= \frac{1}{2} m_{\alpha\beta}(q_\alpha) \dot{\eta}_\alpha \dot{\eta}_\beta \\ &= \frac{1}{2} m_{\alpha\beta}(q_{\alpha,0}) \dot{\eta}_\alpha \dot{\eta}_\beta + O(\eta_\alpha^2) \\ &= \frac{1}{2} \underline{\dot{\eta}}^t \underline{\underline{m}}\, \underline{\dot{\eta}} \end{aligned} \tag{5.5}$$

where $\underline{\underline{m}}$ is the matrix with components $m_{\alpha\beta}(q_{\alpha,0})$. $\underline{\underline{m}}$ is a symmetric real positive matrix. $\underline{\underline{V}}$ is a real symmetric matrix.

We have

$$L = \frac{1}{2} (\underline{\dot{\eta}}^t \underline{\underline{m}}\, \underline{\dot{\eta}} - \underline{\eta}^t \underline{\underline{V}}\, \underline{\eta}) \tag{5.6}$$

The Euler–Lagrange equation (2.8) gives

$$\underline{\underline{m}}\ddot{\eta} = -\underline{\underline{V}}\eta \tag{5.7}$$

It's N coupled linear second order differential equations with constant coefficients (in components of $m_{\alpha\beta}\ddot{\eta}_\beta = -V_{\alpha\beta}\eta_\beta$).

Solutions of (5.7): we try a solution of the form

$$\underline{\eta} = \underline{A}\,\mathrm{e}^{i\omega t} \tag{5.8}$$

Notice that ω is α independent: each η_α is assumed to oscillate with the same frequency ω.

So substitude (5.8) in (5.7). We have $\ddot{\underline{\eta}} = -\omega^2\underline{\eta}$, so

$$\omega^2\underline{\underline{m}}\underline{A} = \underline{\underline{V}}\underline{A} \tag{5.9}$$

A "generalized" eigen-value equation, where $\omega^2 \equiv \lambda$ is the eigen-value and $\underline{A}$ is (not quite) the eigen-vector.

Solution only if

$$\det(\omega^2\underline{\underline{m}} - \underline{\underline{V}}) = 0 \tag{5.10}$$

It is an N^{th} order differential equation in ω^2, with generally N different solutions for ω^2.

Is $\omega^2 > 0$ (oscillatory solutions) or $\omega^2 < 0$ (expontential growth)? Consider (5.9), and multiply on the left by $\underline{A}^\dagger$

$$\omega^2(\underline{A}^\dagger\underline{\underline{m}}\,\underline{A}) = (\underline{A}^\dagger\underline{\underline{V}}\,\underline{A})$$
$$\omega^2 = \frac{\underline{A}^\dagger\underline{\underline{V}}\,\underline{A}}{\underline{A}^\dagger\underline{\underline{m}}\,\underline{A}}$$

since $\underline{\underline{V}}$ and $\underline{\underline{m}}$ are hermitian, both $\underline{A}^\dagger\underline{\underline{V}}\,\underline{A}$ and $\underline{A}^\dagger\underline{\underline{m}}\,\underline{A}$ are real, so $\omega^2 \in \mathbb{R}$.

How about the sign of ω^2? $\underline{A}^\dagger\underline{\underline{V}}\,\underline{A}$ can be positive or negative, depending the sign of (5.2) ($\forall\alpha, \beta, V_{\alpha\beta} > 0$). The equilibrium is:

– stable, $\omega^2 > 0$;
– partially stable if some of $V_{\alpha\beta} < 0$;
– unstable if all $V_{\alpha\beta} < 0$, and $\omega^2 < 0$.

The general solution of (5.7) is

$$\underline{\eta} = \Re\left(\sum_{n=1}^{n} c_n(\underline{A}_n\,\mathrm{e}^{i\omega_n t})\right) \tag{5.11}$$

5.2 Exemples

5.2.1 Normal modes of a linear triatomic molecule

Three atoms (of mass m, M and m) are assumed to be aligned along the same axis, and we only consider motion along that axis. Also assume nearest neighbour interactions

$$L = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2) - V(x_2 - x_1) - V_1(x_3 - x_2) \tag{5.12}$$

Let $x_{i,0}$ be the equilibrium positions of the atoms with

$$x_{3,0} - x_{2,0} = x_{2,0} - x_{1,0} = r_0$$

Denote $x_i(t) = x_{i,0} + \eta_i(t)$ departures from equilibrium positions.

Expand the Lagrangian to second order in η_i

$$V(x_2 - x_1) = \underbrace{\frac{1}{2} \left. \frac{\partial^2 V}{\partial r^2} \right|_{r_0}}_{\equiv k} (\eta_2 - \eta_1)^2$$

where $r = x_2 - x_1 = r_0 + \eta_2 - \eta_1$, and $k > 0$.

Equations of motion are

$$m\ddot{\eta}_1 = -k(\eta_1 - \eta_2)$$
$$M\ddot{\eta}_2 = -k\Big((\eta_2 - \eta_1) + (\eta_2 - \eta_3)\Big)$$
$$m\ddot{\eta}_3 = -k(\eta_3 - \eta_2)$$

Substituting $\eta_i = A_i \, \mathrm{e}^{i\omega t}$

$$\omega^2 \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Solution only if

$$\det \begin{pmatrix} \omega^2 m - k & 1 & 0 \\ 1 & \omega^2 M - 2k & 1 \\ 0 & 1 & \omega^2 m - k \end{pmatrix} = 0$$

Solutions are

– $\omega^2 = 0$ and

$$\underline{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

because

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

– $\omega^2 = k/m$ and

$$\underline{A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

–

$$\omega^2 = k \frac{M + 2m}{Mm}$$

and

$$\underline{A} = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

The general vibrational motion of the molecule is a sum of these three normal modes

$$\begin{aligned}\underline{\eta} =& \Re\Bigg[(C+Dt)\begin{pmatrix}1\\1\\1\end{pmatrix} + E\begin{pmatrix}1\\0\\-1\end{pmatrix}\exp\left(i\sqrt{\frac{k}{m}}\,t\right)\\ &+ F\begin{pmatrix}1\\-2m/M\\1\end{pmatrix}\exp\left(i\sqrt{k\frac{M+2m}{Mm}}\,t\right)\Bigg]\end{aligned}$$

5.2.2 The double pendulum

Two massless rods of length ℓ. The extremity of one is fixed at $(0,0)$, and has a mass m_1, at the other end. The extremity of the other is fixed at m_1, and has a mass m_2 at the other end.

1. Write down the Lagrangian.
2. Find the equilibrium positions.
3. Find the normal modes about the stable equilibrium.

Kinetic energy:

– mass 1:

$$\begin{aligned}x_1 &= \ell\sin\theta_1\\ y_1 &= -\ell\cos\theta_1\\ T &= \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)\end{aligned}$$

– mass 2:

$$\begin{aligned}x_2 &= \ell(\sin\theta_1+\sin\theta_2)\\ y_2 &= -\ell(\cos\theta_1+\cos\theta_2)\\ T &= \frac{1}{2}m_1\ell^2\dot{\theta}_1^2 + \frac{1}{2}\ell^2 m_2\Big(\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\theta_2\cos(\theta_1-\theta_2)\Big)\end{aligned}$$

Potential energy:

$$V = -m_1 g\ell\cos\theta_1 - m_2 g\ell(\cos\theta_1+\cos\theta_2)$$

Equilibirum positions when

$$\frac{\partial V}{\partial\theta_1} = 0 \qquad \frac{\partial V}{\partial\theta_2} = 0$$

so

$$\begin{aligned}\frac{\partial V}{\partial\theta_1} &= (m_1+m_2)\sin\theta_1 = 0\\ &\Longleftrightarrow \theta_1 = 0 \quad\text{or}\quad \theta_1 = \pi\end{aligned}$$

and

$$\frac{\partial V}{\partial\theta_2} = m_2 g\ell\sin\theta_2 = 0 \Longleftrightarrow \theta_2 = 0 \quad\text{or}\quad \theta_2 = \pi$$

There are four equilibrium positions

$$(\theta_1, \theta_2) = \begin{cases} (0,0) & \text{stable} \\ (0,\pi) & \text{partially stable} \\ (\pi,0) & \text{partially stable} \\ (\pi,\pi) & \text{unstable} \end{cases}$$

At the stable position, we have $\theta_1 = \theta_2 = 0$. Denote the perturbations about the stable equilibrium by $\eta_1(t)$ and $\eta_2(t)$, and expand the Lagrangian to the second order in the η_i.

$$\begin{aligned} L = &\frac{1}{2}m_1\ell^2\dot{\eta}_1^2 + \frac{1}{2}m_2\ell^2(\dot{\eta}_1^2 + \dot{\eta}_2^2 + 2\dot{\eta}_1\dot{\eta}_2) \\ &- \Big(\frac{m_1 g\ell}{2}\,\eta_1^2 + \frac{m_2 g\ell}{2}(\eta_1^2 + \eta_2^2)\Big) \end{aligned}$$

The Euler–Lagrange equations for η_i are in matrix form

$$\ell^2 \begin{pmatrix} M & m \\ m & m \end{pmatrix} \begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{pmatrix} = g\ell \begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where $M = m_1 + m_2$ and $m = m_2$.

Set $\eta_i = A_i\,\mathrm{e}^{i\omega t}$, solve

$$\det(\omega^2 \underline{\underline{m}} - \underline{\underline{V}}) = 0$$

to find

$$\omega^2 = \frac{g}{\ell}\frac{M}{m}\left(1 \pm \sqrt{\frac{m}{M}}\right)$$

with eigen-vectors

$$A = \begin{pmatrix} \sqrt{m} \\ \pm\sqrt{M} \end{pmatrix}$$

Chapter 6

Constrained systems and Lagrange multipliers

So far, we have always assumed that the generalized coordinates q_α were independent. We will now consider situations in which the q_α are not longer independent. In this case, the Euler–Lagrange equations (2.8) are not valid.

Recall the example of the particle moving on the wire of shape $y = Ax^2$. We used x as the single generalized coordinates, and then used the Euler–Lagrange to determine the dynamics of the particle. If you decide to use x *and* y as generalized coordinates, these would no longer be independent as $y = Ax^2$, and then you can not use the Euler–Lagrange equations.

When deriving the Euler–Lagrange equations, we minimized the action and arrived at the equation

$$\delta S = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\alpha} \right) \delta q_\alpha \mathrm{d}t = 0$$

For example, if $\alpha = 1, 2$, then

$$\delta S = \int_{t_i}^{t_f} \left[\left(\frac{\partial L}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_1} \right) \delta q_1 + \left(\frac{\partial L}{\partial q_2} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_2} \right) \delta q_2 \right] \mathrm{d}t = 0$$

Only if δq_1 and δq_2 are independent can we conclude from that

$$\frac{\partial L}{\partial q_1} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_1} \qquad \frac{\partial L}{\partial q_2} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_2}$$

If δq_1 and δq_2 are related then you can not obtain the Euler–Lagrange equations.

There are different types of constraints:

– *holonomic* constraints: These are constraints of the form

$$h_i(q_1, \ldots, q_N) = 0 \qquad i \in [\![1, n]\!] \tag{6.1}$$

where n is the number of holonomic constraints, so there are $N - n$ independent coordinates.

Exemple 6.1.
Read on the wire of shape $y = Ax^2$. If one solves this problem using both x and y as generalized coordinates, then these are related by the holonomic constraints

$$h(x, y) = y - Ax^2 = 0$$

Exemple 6.2.
Cylinder of radius a rolling down a shape. Here y and ϕ are obvious generalized coordinates, but they are linked by the holonomic constraints

$$h = y - a\phi = 0$$

Notice

$$\delta h_i = \sum_{\alpha=1}^{N} \frac{\partial h_i}{\partial q_\alpha} \, \delta q_\alpha = 0$$

– *non-holonomic* constraints: these come in different forms
  – differential constraints between the generalized coordinates

$$\sum_{\alpha=1}^{N} a_{i\alpha} \mathrm{d}q_\alpha + b_i \mathrm{d}t = 0 \tag{6.2}$$

  which can not be integrated.
  – inequalities.
  Exemple 6.3.
  A ball on a curcular table of radius a. You would need to impose $x^2 + y^2 \leq a^2$.

6.1 Lagrange multipliers

Suppose we make a profit of $G(x, y, z)$ euros when we sell a cereal bar with x ,y, z grams of additives. But european regulations impose that x, y, z must satisfy

$$h(x, y, z) = 0$$

where h is a given function.

The aim is to manufacture a cereal bar which
– the (x, y, z) lie on the 2-dimensional surface $h = 0$.
– on this surface you want to maximize your profit.

Consider small variations of the inputs x, y, z. Then G varies by

$$\delta G = \frac{\partial G}{\partial x} \, \delta x + \frac{\partial G}{\partial y} \, \delta y + \frac{\partial G}{\partial z} \, \delta z = 0$$

but since $h(x, y, z) = 0$, we also have

$$\delta h = \frac{\partial h}{\partial x} \, \delta x + \frac{\partial h}{\partial y} \, \delta y + \frac{\partial h}{\partial z} \, \delta z = 0$$

Hence $\lambda h = 0$ also, where $\lambda(x, y, z)$ (Lagrange multiplier) is an arbitraty function. Then we have

$$\delta G - \lambda \delta h = 0$$
$$\left(\frac{\partial G}{\partial x} - \lambda \frac{\partial h}{\partial x}\right) \delta x + \left(\frac{\partial G}{\partial y} - \lambda \frac{\partial h}{\partial y}\right) \delta y + \left(\frac{\partial G}{\partial z} - \lambda \frac{\partial h}{\partial z}\right) \delta z = 0$$

We only have two independent variables (x and y) since $h = 0$ impose the relationship. Let use choose the arbitraty function λ such that

$$\frac{\partial G}{\partial z} - \lambda \frac{\partial h}{\partial z} = 0 \tag{6.3}$$

and since x and y are independent, equations which must be satisfied are

$$\left(\frac{\partial G}{\partial x} - \lambda \frac{\partial h}{\partial x}\right) \delta x = 0$$
$$\left(\frac{\partial G}{\partial y} - \lambda \frac{\partial h}{\partial y}\right) \delta y = 0$$

To solve the problem, you need to solve these equations plus the constraint equation $h = 0$.

Let's return to the action S which we want to minimize subject to the constraint $h = 0$. Since both $\delta S = 0$ and $\delta h = 0$, consider

$$\delta S - \int_{t_i}^{t_f} \mathrm{d}t (\lambda \delta h) = 0 \tag{6.4}$$

where λ is an arbitraty function of (q_α, t).

$$\int_{t_i}^{t_f} \mathrm{d}t \sum_{\alpha=1}^{N} \delta q_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} - \lambda \frac{\partial h}{\partial q_\alpha}\right) = 0 \tag{6.5}$$

There are $N - 1$ independent coordinates in the problem, which we take to be $(q_1, \ldots, q_{N-1})$. Let's choose λ such that for $\alpha = N$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_N} - \frac{\partial L}{\partial q_N} = \lambda \frac{\partial h}{\partial q_N}$$

Then (6.5) becomes

$$\int_{t_i}^{t_f} \mathrm{d}t \sum_{\alpha=1}^{N-1} \delta q_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} - \lambda \frac{\partial h}{\partial q_\alpha}\right) = 0$$

But the $(q_1, \ldots, q_{N-1})$ are independent generalized coordinates so

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = \lambda \frac{\partial h}{\partial q_\alpha} \qquad \alpha \in [\![1, N-1]\!] \tag{6.6}$$

plus the equation $h = 0$.

If you have $(\lambda_1, \ldots, \lambda_n)$ constraints, one simply replaces

$$\lambda \frac{\partial h}{\partial q_\alpha} \longrightarrow \sum_{i=1}^{n} \lambda_i \frac{\partial h}{\partial q_\alpha} \tag{6.7}$$

One can show that the Lagrange multipliers have a physical meaning. They must be related to the external forces which are imposing the constraint between the generalized coordinates. Indeed

$$F_\alpha^{\text{constraint}} = \lambda \frac{\partial h}{\partial q_\alpha} \tag{6.8}$$

Exemple 6.4 (Cylinder rolling down an inclined plane)**.**
Initially the cynlinder is at rest at $y = 0$. Find the Lagrangian, the Lagrange multipliers, forces on the cylinder and solve the problem.
Use as generalized coordinates y and ϕ which are linked through

$$h = y - a\phi$$

The Lagrangian is

$$L = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I\dot{\phi}^2 + mgy \sin\alpha$$

The Euler–Lagrange equations with the constraints are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda \frac{\partial h}{\partial y}$$
$$\Longrightarrow m\ddot{y} - mg\sin\alpha = \lambda$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \lambda \frac{\partial h}{\partial \phi}$$
$$\Longrightarrow I\ddot{\phi} = -a\lambda$$

and constraint equation $y = a\phi$.
With $I = ma^2/2$

$$\frac{1}{2} ma^2 \frac{\ddot{y}}{a} = -a\lambda$$
$$\Longrightarrow \frac{1}{2} m\ddot{y} = -\lambda$$

Eliminate λ in the first equation

$$\ddot{y} = \frac{2}{3} g \sin\alpha$$

Integrate and use initial conditions

$$\boxed{y = \frac{1}{3} gt^2 \sin\alpha}$$

hence from the third equation

$$\boxed{\phi = \frac{y}{a} = \frac{1}{3a} gt^2 \sin\alpha}$$

and

$$\boxed{\lambda = \frac{-1}{3} gm \sin\alpha}$$

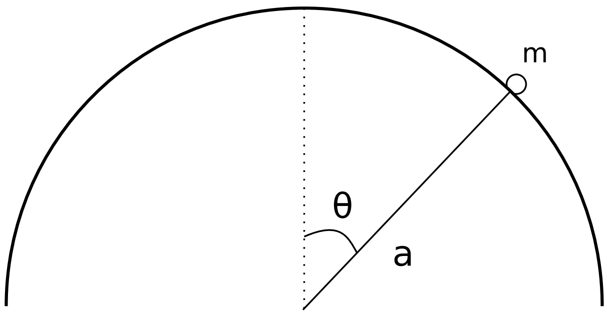

Figure 6.1: Particle on an hemisphere.

Finally the constraint forces on the cylinder (making it roll rather than slip) are

$$F_y^{\text{constraint}} = \frac{-1}{2} gm \sin\alpha$$
$$F_\phi^{\text{constraint}} = \frac{a}{3} gm \sin\alpha$$

Exemple 6.5.

A particle of mass m starts at rest on the top of a smooth fixed hemisphere of radius a (figure **??**). Find the constraint force, and determine when the particle leaves the hemisphere.

Choose generalized coordinates (r, θ). There are not independent when the particle is on the hemisphere, since then $r = a$.

It's an holonomic constraint : $h = r - a = 0$. The Lagrangian in terms of (r, θ) is

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

The Euler–Lagrange equations with constraints are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda \frac{\partial h}{\partial r}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial h}{\partial \theta}$$

so we need to solve

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda \\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) - mgr\sin\theta = 0 \\ h = 0 \Longleftrightarrow r = a \end{cases}$$

From the third we deduce $\dot{r} = 0$, so

$$ma\dot{\theta}^2 - mg\cos\theta = \lambda$$
$$\ddot{\theta} - \frac{g}{a}\sin\theta = 0$$

Aim is to find the constraint force and hence we need to find λ. To do so multiply the previous equation by $\dot{\theta}$ and integrate

$$\frac{\dot{\theta}^2}{2} + \frac{g}{a}\cos\theta = K$$

But initially the particle is at rest, at $\theta = 0$ ($\dot{\theta} = 0$) so $K = g/a$. Hence

$$\frac{\dot{\theta}^2}{2} + \frac{g}{a}\cos\theta = \frac{g}{a}$$
$$a\dot{\theta}^2 - g\cos\theta = \frac{-\lambda}{m}$$

Eleminate $\dot{\theta}^2$ between these equations:

$$\lambda = mg(3\cos\theta - 2)$$

and constraint force is $F_\alpha = \lambda \partial_{q_\alpha} h$. The particle falls off the hemisphere when there is no force to hold on it, so when $\lambda = 0$:

$$\cos\theta = \frac{2}{3}$$

Chapter 7

From Lagrangians to Hamiltonians – The Hamiltonian formulation of classical mechanics

Lagrangian approach:

- Lagrangian $L(q_\alpha, \dot{q}_\alpha, t)$
- N independent coordinates $q_\alpha \rightarrow N$-dimensional configuration space
- Euler–Lagrange equations : N second order equations
- To solve these equations, one need $2N$ initial/boundary conditions
- Equations invariant under $q_\alpha \rightarrow Q(q_1, \dots, q_N, t)$

Hamiltonian approach:

- Hamiltonian $H(q_\alpha, p_\alpha, t)$
- $2N$ dimensional *phase space* spanned by (q_α, p_α)
- $2N$ first order equations
- $2N$ boundary conditions
- Equations are invariant under a much larger set of "canonical transformation" $q_\alpha \rightarrow Q(q_1, \dots, q_N, t)$ and $p_\alpha \rightarrow P(p_1, \dots, p_N, t)$

The Hamiltonian formulation is not particularly better than the Lagrangian are for solving the problems we have studied so far. But the Hamiltonian formulation provides links to many other differents branches of physics :

- H and L are linked by a Legendre transformation $\longleftrightarrow$ thermodynamics (energy E and enthalpy H).
- Phase space dynamics $\longleftrightarrow$ chaos, QM...
- Poisson brackets $\longleftrightarrow$ commutators in quantum mechanics (groups, Lie algebras...).

The Hamiltonian approach is very powerful but not adapted to the study of relativistic systems (particle physics, classical field theory, general relativity...). Here you need to use the Lagrangian which can be written in a "covariant" way.

As a mathematical problem, the transformation from L to H is a change of variables from $L(q_\alpha, \dot{q}_\alpha, t)$ to $H = H(q_\alpha, p_\alpha, t)$, where

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha} \tag{7.1}$$

called the generalized momenta.

Exemple 7.1.
Consider a function $f(x, y)$ with differential

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \equiv u\mathrm{d}x + y\mathrm{d}y$$

Suppose we want to change the basis of our description from the variables (x, y) to (u, y). In terms of (u, y) differentials are expressed in terms of du and dy.
Let $g = f - ux$ (Legendre transformation). Then

$$\begin{aligned} \mathrm{d}g &= \mathrm{d}f - u\mathrm{d}x - x\mathrm{d}u \\ &= y\mathrm{d}y - x\mathrm{d}u \end{aligned}$$

so that $g = g(u, y)$.

Going from L to H is an identical procedure:

$$H(p_\alpha, q_\alpha, t) = p_\alpha \dot{q}_\alpha - L$$

What are the Hamiltons equations? Calculate the differential of H

$$\begin{aligned} \mathrm{d}H &= p_\alpha \mathrm{d}\dot{q}_\alpha + \dot{q}_\alpha \mathrm{d}p_\alpha - \mathrm{d}L \\ &= \dot{q}_\alpha \mathrm{d}p_\alpha + \frac{\partial L}{\partial q_\alpha}\mathrm{d}q_\alpha + \frac{\partial L}{\partial t}\mathrm{d}t \\ &= \frac{\partial H}{\partial p_\alpha}\mathrm{d}p_\alpha + \frac{\partial H}{\partial q_\alpha}\mathrm{d}q_\alpha + \frac{\partial H}{\partial t} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial H}{\partial q_\alpha} &= -\frac{\partial L}{\partial q_\alpha} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} = -\dot{p}_\alpha \end{aligned}$$

hence

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \tag{7.2a}$$

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \tag{7.2b}$$

Remembering Beltrami (3.2):

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} \tag{7.3}$$

H is constant if L is explicitly time independent, and corresponds to the total energy in certain cases.

Exemple 7.2.

A particule of mass m in a central potential $V(r)$. Work in spherical polar coordinates (r, θ, ϕ).

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\phi^2 + r^2\dot{\theta}^2) - V$$
$$p_r = m\dot{r} \qquad p_\phi = mr^2\phi\sin^2\theta \qquad p_\theta = mr^2\dot{\theta}$$

Replace in

$$\begin{aligned} H &= p_\alpha \dot{q}_\alpha - L \\ &= p_r\dot{q}_r + p_\phi\dot{q}_\phi + p_\theta\dot{q}_\theta - L \\ &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V \end{aligned}$$

7.1 Poisson brackets

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} \\ &= \left(-\frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p}\right) + \frac{\partial f}{\partial t} \\ &= [f, H]_{PB} + \frac{\partial f}{\partial t} \end{aligned}$$

where the Poisson bracket of two function $f(q, p, t)$ and $g(q, p, t)$ is

$$[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{7.4}$$

You can check that

$$[f, g] = -[g, f]$$
$$[f + g, h] = [f, h] + [g, h]$$
$$\Big[[f, g], h\Big] + \Big[[g, h], f\Big] + \Big[[h, f], g\Big] = 0 \qquad \text{Jacobi identity}$$
$$[p, q] = 1 \qquad [p, p] = [q, q] = 0$$

The link between classical mechanics and quantum mechanics is made by:

– replacing allfunctions by operators:

$$f \longrightarrow \hat{f} \qquad H \longrightarrow \hat{H}$$

–

$$[f, g] \longrightarrow \frac{-i}{\hbar}[\hat{f}, \hat{g}] \qquad \text{commutators}$$
$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = i\hbar[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\mathrm{d}t}$$

Part II

Special relativity

Chapter 8

Introduction

The fundamental ingredients of special relativity are

– the speed of light c, which is a constant of nature ;
– nothing (no signal, electromagnetic wave, information...) car propagate faster than c.

These are encoded into the two postulates on which special relativity is based (Einstein, 1905).

Postulat 8.1 (Principle of relativity)**.** The laws of physics are the same in inertial frames. In other words, there is no preferred rest frame.

Postulat 8.2 (Universality of the speed of light)**.** The speed of light is the same in all inertial frames.

What are the consequences?

1. Our intuitions on relative speeds is wrong.

   Exemple 8.1.

   Suppose we are sitting in a rocket which has a constant velocity v, relative to the earth's frame (which, for the purposes of the example, is taken to be an inertial frame).

   Suppose that a light signal is emitted from the back of the rocket, outside.

   According to an observer on earth, the light signal propagates with speed c. According to our classical mechanics intuition, the observer in the rocket should measure the speed of light signal to be $c' = c - v$.

   Conversly: if you measure c' from within the rocket, et c outside, then you should be able to determine v.

   But all experiments of this kind have always given $c' = c$, hence the origin of the second postulate of special relativity.

   This result is in complete agreement with Maxwell's equations in vacuum.

   Special relativity resolves the contraction, which reigned in physics at the time of Maxwell, Lorentz, Minkowski, Einstein.

2. Galilean transformations must be wrong.

   Consider two reference frames $\mathcal{R}$ and $\mathcal{R}'$, where $\mathcal{R}'$ moves at constant speed v relative to $\mathcal{R}$ along their common x-axis. Galilean transformations are

$$x' = x - vt$$
$$y' = y \qquad z' = z$$
$$t' = t$$

They are wrong because they give $c' = c - v$ which is incompatible with experiment. They must be replaced by the Lorentz transformations

$$x' = \gamma(v)(x - vt)$$
$$y' = y \qquad z' = z$$
$$t' = \gamma(v)\left(t - \frac{vx}{c^2}\right)$$

where $\gamma(v)$ is the Lorentz factor

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Note that coordinates y and z which are perpendicular to the velocity are unchanged.

In non-relativistic limit, $v/c \ll 1$, the Lorentz transformations reduce to the Galilean transformations.

3. Loss of simultaneity: if two events happen at the same time in one reference frame, they occur at different times in other reference frame.
4. Lengths are not constant: the lengths measures depend on the frame you are in (it's not a scalar).
5. Newton's law must be replaced by

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(v)m\boldsymbol{v})$$

6. Space and time are no longer separated: one must work in 4-dimensional space-time.
7. The famous equation

$$E = mc^2$$

or more exactly $E = \sqrt{p^2c^2 + m^2c^4}$.

Remarque : In special relativity, space-time is non-dynamical. In general relativity, space-time is dynamical.

Chapter 9

The postulates of special relativity and the Michelson–Morley experiment

The first postulate 8.1 can be reworded in different ways:

– all laws of physics are the same in inertial frames;
– no experiment can measure the absolute velocity of an observer: the result of all experiments made by an observer is independent of his speed relative to other observers not involved in the experience;
– there is no special/privileaged frame or observer from the point of view of physics: the vacuum is not absolute.

Définition 9.1 (Frame)**.** A frame is an ensemble of observers who are at rest relative to eachother, and each of wchich has a clock. All the clocks are synchonised.

Définition 9.2 (Inertial frame)**.** An inertial frame is one in which, in the absence of forces, a particle moves with constant velocity $\boldsymbol{v}$.

There is an infinite number of inertial frames.

In non-relativistic physics, the principle of relativity is encoded in the invariance of physical laws and equations under Galilean transformations. Given a frame $\mathcal{R}(t, x, y, z)$ and a frame $\mathcal{R}'(t', x', y', z')$ moving with constant veolocity u along teir common x-axis, then the Galilean transformations are

$$\begin{gathered} x' = x - ut \\ y' = y \qquad z' = z \qquad t' = t \\ \Longrightarrow \boldsymbol{v'} = \boldsymbol{v} - \boldsymbol{u} \qquad a' = a \end{gathered}$$

Maxwells equations in SI[1] units, and in the vacuum are

$$\nabla \boldsymbol{E} = 0 \tag{9.1a}$$
$$\nabla \boldsymbol{B} = 0 \tag{9.1b}$$
$$\nabla\times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t} \tag{9.1c}$$
$$\nabla\times \boldsymbol{B} = \mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{9.1d}$$

These can be combined, giving

$$\left(\frac{\partial^2}{\partial t^2} - c^2\,\nabla^2\right)\boldsymbol{E} = 0 \tag{9.2a}$$
$$\left(\frac{\partial^2}{\partial t^2} - c^2\,\nabla^2\right)\boldsymbol{B} = 0 \tag{9.2b}$$

with

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{9.3}$$

the speed of light in vacuum.

Maxwells equations (9.1) are written in an inertial frame and hence c is the speed of light in that frame.

Contrary to Newtons equations, the equations change under a Galilean transformations, one obtains for the wave equations (9.2)

$$\left(\frac{\partial}{\partial t'} + (c-u)\frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t'} - (c+u)\frac{\partial}{\partial x'}\right)E' = 0$$

Equations consisting of waves propagating with velocity $c' = c \pm u$. This equation is not (9.2), so Maxwells equations are not invariant under Galilean transformations.

As we discussed, experiments were designed to measure "our" speed relative to a prefered inertial frame in which light waves propagate with veolocity c', but one found $c' = c$.

What to do?

1. Perhaps Maxwell's equations are wrong, and need to be modified in order to make Galilean invariant? Lots of trials, but all such modification lead to new electrical phenomena which are ruled experimentally.
2. Perhaps Maxwell's equations are only valid in a priveledged "ether frame"? And hence in a different inertial frame, light travels with $c' = c \pm u$.
3. Perhaps the Galilean transformations are wrong and Maxwell's equations are true in all inertial frames? In this case, one should replace the Galilean transformations by other transformations, which will then no longer leave Newtons laws invariant.

1. Standard international.

9.1 The Michelson–Morley interferometer

Aim was to measure c' and hence find our velocity relative to the "ether" frame. At the time, the largest velocity accessible was that of the earth around the sun (30 km/s).

- If the experiment is at rest in the ether frame, the times are equal, and hence the beams arrive on the screen in phase.
- Now suppose that the experiment moves with speed u relative to the ether frame. Let's analyse it in the ether frame, where the speed of light is always c.
  Let t_1 be the time for light to go from B' to C'. In this time, light travels a distance $L_1 + ut_1$ (emitted at B and arrives at C' because C has moved). But we are working in the ether frame where light travels at speed c. Hence $L + ut_1 = ct_1$ and

$$t_1 = \frac{L}{c-u} \tag{9.4}$$

Let t_2 be the time taken for light to go from C' back to B' ($d = L - ut_2$)

$$t_2 = \frac{L}{c+u} \tag{9.5}$$

Hence

$$\begin{aligned}\Delta t_{\parallel} &= t_1 + t_2 \\ &= \frac{L}{c-u} + \frac{L}{c+u} \\ &= \frac{2Lc}{c^2 - u^2} \\ &\boxed{= \frac{2L/c}{1 - v^2/c^2}}\end{aligned} \tag{9.6}$$

Let t_3 be the time taken for light to go from B' to D'

$$\begin{aligned}\sqrt{L^2 + (ut_3)^2} &= ct_3 \\ t_3^2(c^2 - u^2) &= L^2\end{aligned}$$

so

$$\boxed{t_3 = \frac{L/c}{\sqrt{1 - u^2/c^2}}} \tag{9.7}$$

Let t_4 be the time to go back. It is also t_3 (symmetry).

$$\begin{aligned}\Delta t_{\perp} &= t_3 + t_4 \\ &\boxed{= \frac{2L/c}{\sqrt{1 - u^2/c^2}}}\end{aligned} \tag{9.8}$$

On a $\Delta t_{\perp} \neq \Delta t_{\parallel}$. Hence the two beams arrive at the screen with phase different

$$\begin{aligned}\Delta\phi &= \omega(\Delta t_{\parallel} - \Delta t_{\perp}) \\ &\boxed{= 2\pi \frac{L}{\lambda} \frac{u^2}{c^2} + \theta \frac{u^4}{c^4}}\end{aligned} \tag{9.9}$$

which is sufficiency big to be measured ($8u \approx 30\,\text{km/s}$: but no effect has ever been measured.

Chapter 10

Consequences of the constancy of c

About 18 years after the Michelson–Morley experiment, Einstein postulates the second principle of relativity 8.2.

Corollaires: Galilean transformations have to be replaced by a new set of transformations that leave Maxwell equations invariant.

10.1 Space-time diagram

A space-time diagram is the (t, x) plane (we suppress y and z because we can not draw in four dimensions).

From now, we set $c = 1$: t and x are both measured in meter.

Définition 10.1 (Event)**.** An event is an event which happens at a given value of (t, x, y, z). Hence it's a point on a space-time diagram.

Définition 10.2 (World-line)**.** A world-line of a particle is a curve $x(t)$, which gives the position of the particle as a function of time.

We have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{v(t)}$$

Since no information can travel faster than light

$$v(t) \leq 1 \Longrightarrow \frac{\mathrm{d}t}{\mathrm{d}x} \geq 1$$

The slope of the world line of a photon is ± 1.

Careful: this is a space-time diagram: $\sqrt{x_e^2 + t_e^2}$ is not the "distance" between the origin and the event. You can not use Pythagoras theorem, etc., on these space-time diagram.

Since $v \leq 1$, the concepts of future and past are modified with respect to Newtonian physics. In Galilean physics, space-time is the union of the future (everything that occures after a giving event) and the past (before that event).

It is no longer valid in special relativity. The space-time diagram is divided in three areas.

Now we want to include on the space-time diagram of O a second observer O' moving with constant velocity u along their common axis x. Suppose that at $t = x = 0$, $t' = x' = 0$.

We now need to construct the x' axis (set of all events which occurs at $t' = 0$). Let's consider the space-time diagram of O'. All events on the x' axis have the following property: a light ray is emitted at $x' = 0$ and $t' = -a$, and arrives on the x' axis at $t' = 0$ and $x' = a$. If it's reflected, it returns to the t' axis at $t' = 0$.

Let's try to draw this on the space-time diagram of O.

Remarque : On a space-time diagram, x' and x axis do not coincide. And t' and x' are not perpendicular. The angle θ is given by

$$\tan\theta = \frac{x}{t} = u \tag{10.1}$$

10.2 Simultaneity

Consider two events A and B.

$$(t_A, x_A) = (0, x_A) \qquad (t_B, x_B) = (0, 0)$$

These events are simultaneous for the observer O because they both occur at $t = 0$.

For observer O', $t'_B = 0 = x'_B$, and $t'_A < 0$. The two events are not simultaneous since $t'_A < t'_B$.

In special relativity, one therefore talks of the "relativity of simultaneity", i.e simultaneity is observer-dependent.

10.3 Invariant interval

Consider E and P on the world line of a photon. Both observer agree that

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 = 0$$
$$(\Delta s')^2 = (\Delta t')^2 - (\Delta x')^2 = 0$$

because of the speed of light. So if $\Delta s = 0$ in one frame then $\Delta s' = 0$ in the other.

Now consider any two arbitrary events B and Q: $t_B = x_B = 0$ in O, and (t_Q, x_Q) in O, $t'_B = x'_B = 0$, and (t'_Q, x'_Q) in O'.

We define the invariant interval by

$$\boxed{(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} \tag{10.2}$$
$$(\Delta s')^2 = (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

One can show that

$$\Delta s = \Delta s' \tag{10.3}$$

i.e. Δs is an invariant on which all observers agree.

Proof.

- First note that if Q and O are on a photon worldline then $\Delta s = \Delta s' = 0$.
- More generally, we use as basic input that space and time are assumed homogeneous [1] and that space is isotropic [2].

□

Careful: Δs^2 is not necessary positive (think with $\Delta t^2 = 0$).

The consequence of this is that the transformations between (x', t') and (x, t) must be linear:

$$\begin{cases} x' = \alpha x + \beta t \\ t' = \gamma x + \delta t \end{cases}$$

Hence

$$\begin{aligned}(\Delta s')^2 &= (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= t_Q'^2 - x_Q'^2 \\ &= Q(t, x, y, z)\end{aligned}$$

where

$$Q(t, x, y, z) = \sum_{i=1}^{3} (A_{ij} x^i x^j + B_i x^i t + ct^2)$$

since the transformation is linear, and the coefficient A_{ij}, B_i and C are all functions of $\|\boldsymbol{u}\| = u$ by isotropy, and $(x^1, x^2, x^3) = (x, y, z)$.

We know that $Q = 0$ on a photon world line, i.e. when $t_\ell^2 = x^2 + y^2 + z^2$ (photon at c, $t^2 = x^2$ in one dimension). Hence $Q(t_\ell, x, y, z) = Q(t_\ell, -x, -y, -z) = 0$ meaning that $B_i = 0$ and

$$\sum A_{ij} x^i x^j + C t_\ell^2 = 0$$

Hence:

$$\begin{aligned}(\Delta s')^2 &= -C t_\ell^2 + C t^2 \\ &= -C(x^2 + y^2 + z^2) + C t^2 \\ &= C(u) \Delta s^2\end{aligned}$$

We want to show that $c = 1$. We have observers:

- O.
- O' moving with velocity u relative to O.
- O'' moving with velocity v relative to O and velocity w relative to O'.

then

$$\begin{aligned}(\Delta s'')^2 &= C(v)(\Delta s)^2 \\ &= C(w)(\Delta s')^2 \\ &= C(w)C(u)(\Delta s)^2 \qquad \Longrightarrow C(v) = C(u)C(w)\end{aligned}$$

with solution $C = 1$. Hence $(\Delta s')^2 = (\Delta s)^2$, all observers agree on the value of Δs.

1. Invariant under translation: there are no prefered position in space or time.
2. Invariant under rotations.

10.3.1 Interval types

The interval is said to be
– time-like if $\Delta s^2 > 0$;
– space-like if $\Delta s^2 < 0$;
– light-like if $\Delta s^2 = 0$.
If $\Delta s^2 \geq 0$, the two events are causally connected since

$$\begin{aligned}\Delta s^2 \geq 0 &\Longrightarrow \Delta t^2 - \Delta x^2 \geq 0\\ &\Longrightarrow 1 - \left(\frac{\Delta x}{\Delta t}\right)^2 \geq 0\\ &\Longrightarrow v^2 \leq 1\end{aligned}$$

There exists a reference frame moving with constant velocity v such that, in this frame, the two events occur at the same spatial position.

If $\Delta s^2 < 0$, the two events are causally disconnected: $v > 1$, so it is impossible.

Remarque : Some authors prefer to use the convention

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

10.4 Calibration of (t', x') axis

How can we callibrate the (t, x') axes on the space-time diagram of O? Where is $t' = 1$ mark?

To do this, we use the invariant interval, taking one event to be the origin $(t = x = t' = x' = 0)$ and the other being a general event (t, x) which has coordinates (t', x') in O'.

$$\Delta s^2 = t^2 - x^2 = t'^2 - x'^2$$

$\Delta t = t - 0$

Which set of events have space-like invariant interval given by $\Delta s^2 = -a^2$?

$$t = \pm\sqrt{x^2 - a^2}$$

Which set of events have time-like invariant interval given by $\Delta s^2 = b^2$?

$$t = \pm\sqrt{x^2 + b^2}$$

If we choose $b = 1$, then, from the invariance of the interval, $t'^2 - x'^2 = 1$. But by definition, on the t' axis, $x' = 0$. Hence the intersection of the t' axis with curve $t^2 - x^2 = 1$ corresponds to $t' = \pm 1$.

10.5 Proper time and time dilatation

For a time-like interval, the proper time $\Delta\tau$ is defined by

$$\Delta s^2 = c^2 \Delta\tau^2 \tag{10.4}$$

What is the physical meaning of τ?

Consider the causally connected events O and $A(t_A, x_A)$. There exists a frame moving woth velocity v in which two different events occur at the same position $x' = 0$ but at different time. Using the invariant interval

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = \Delta t'^2 - \underbrace{\Delta x'^2}_{=0}$$

So, with $c = 1$, τ is nothing other that the time t'. From above

$$\Delta t' = \sqrt{\Delta t^2 - \Delta x^2} \tag{10.5}$$

$$\boxed{= \Delta t\sqrt{1 - v^2}} \tag{10.6}$$

Since $v < 1$, $\Delta t' < \Delta t$. The time that you measure between two events is not the same depending of the observer you are.

Exemple 10.1 (Cosmic rays)**.**

More radioactivity was measured on earth that could actually be explained by known sources of radioactivity. Hess (1912) went up in a balloon to altitudes higher than 6000 m and discovered that radioactivity increases with altitude. He deduced that there were radioactivity sources outside the earth: the cosmic rays.

Cosmic rays consist not only of rays (X- and γ-rays), but mostly charged particles (protons. . .). Their energy are between 10^8 eV ($v \approx 0.4c$ — assuming protons) and 10^{21} eV ($v \approx 0.999c$).

When a cosmic ray interacts with the particles in the upper atmosphere, they produce muons (μ)... Muons decay trhough $\mu^- \to e^- + \nu_e + \nu_\mu$ with $\tau_\mu = 2.2 \times 10^{-5}$ s in the frame in which the muon is at rest.

Without special relativity: the muon travels a distance $d = v \times \tau_\mu = 659\,\text{m} < 12\,\text{km}$, but muons are detected on earth in large quantities. This is because of the time dilatation. On the earth frame, the muon decay is a time $t_\mu = \gamma\tau_\mu$ and $v \times t_\mu \gg 12\,\text{km}$.

So the distance they actually travel aroud 15 km.

Chapter 11

Lorentz transformations

Let's derive them: to do so, we again

– assume homogeneity and isotropy of space-time;

– recall that $c = cste$ only affects the axis parallel to the direction of motion.

As usual, take two frames O and O' with constant velocity v along their common axis x. We can already say that $y' = y$ and $z' = z$. Homogeneity of space and time implies that the transformations are linear:

$$t' = \alpha_1 t + \alpha_2 x x' = \alpha_3 x + \alpha_4 t \tag{11.1}$$

(α_i depend on $|v|$ by isotropy).

Recall that the t' axis has equation $t/x = 1/x$. Hence

$$\begin{aligned} 0 &= \alpha_3 t v + \alpha_4 t \\ 0 &= \alpha_3 v + \alpha_4 \end{aligned}$$

The x' axis has equation $t/x = v$. Hence

$$0 = \alpha_1 v + \alpha_2$$

Hence

$$\begin{aligned} t' &= \alpha_1 (t - vx) \\ x' &= \alpha_3 (x - vt) \end{aligned}$$

Now let's use the invariant interval, which tells us that

$$\begin{aligned} t^2 - x^2 &= t'^2 - x'^2 \\ &= \alpha_1^2 (t - vx)^2 - \alpha_3 (x - vt)^2 \\ &= (\alpha_1^2 - \alpha_3^2 v^2) t^2 + (\alpha_1^2 v^2 - \alpha_3^2) x^2 - 2vxt(\alpha_1^2 - \alpha_3^2) \end{aligned}$$

so

$$\alpha_1 = \alpha_3$$

and

$$\alpha_1^2 (1 - v^2) = 1 \Longrightarrow \alpha_1 = \boxed{\frac{1}{\sqrt{1 - v^2}} \equiv \gamma}$$

The Lorentz transformations are

$$\boxed{\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}} \tag{11.2}$$

The inverse transformations are

$$\begin{aligned} t &= \gamma(t' + vx') \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned} \tag{11.3}$$

But for the other observer, the frame moves with velocity $-v$.
Caution: never use the Lorentz transformations to relate two different events.
In matrix form we have

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{11.4}$$

It's useful to write down the Lorentz transformations for the frame O' moving relating to O in some arbitrary direction $\boldsymbol{v}$. To do so, decompose the spatial position vector $\boldsymbol{r} = (x, y, z)$ of the event into components parallel and perpendicular to $\boldsymbol{v}$.

Let $\boldsymbol{e_v} \parallel \boldsymbol{v}$ a unit vector so that

$$\begin{aligned} \boldsymbol{r}_{\parallel} &= (\boldsymbol{r} \cdot \boldsymbol{e_v})\boldsymbol{e_v} = \underbrace{(\boldsymbol{r} \cdot \boldsymbol{v}\,\boldsymbol{v})}_{=\|\boldsymbol{v}\|^2} \\ \boldsymbol{r}_{\perp} &= \boldsymbol{r} - \boldsymbol{r}_{\parallel} \end{aligned}$$

Under a Lorentz boost in the $\boldsymbol{v}$ direction:

$$\begin{aligned} \boldsymbol{r}_{\perp}' &= \boldsymbol{r}_{\perp} \\ \boldsymbol{r}_{\parallel}' &= \gamma(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t) \end{aligned}$$

so that

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r}_{\perp}' + \boldsymbol{r}_{\parallel}' \\ &= \boldsymbol{r}_{\perp} + \gamma(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t) \\ &= (\boldsymbol{r}_{\perp} + \boldsymbol{r}_{\parallel}) + (\gamma - 1)\boldsymbol{r}_{\parallel} - \gamma\boldsymbol{v}t \\ &= \boldsymbol{r} + (\gamma - 1)\frac{\boldsymbol{r} \cdot \boldsymbol{v}}{v^2}\,\boldsymbol{v} - \gamma\boldsymbol{v}t \end{aligned}$$

In matrix form

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + \frac{(\gamma-1)v_x^2}{v^2} & \ddots & \ddots \\ -\gamma v_y & \ddots & \ddots & \ddots \\ -\gamma v_z & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{11.5}$$

11.1 Time dilatation with Lorentz transformations

Consider two events A and B in frame O, which take place at different t_A and t_B but at the same position $x_A = x_B = x_0$. So the time between the events o, frame O os $\delta t = t_B - t_A$. Now go to a moving frame O'.

$$\begin{aligned}\Delta t' &= t'_B - t'_A \\ &= \gamma(t_B - vx_0) - \gamma(t_A - vx_0) \\ &= \gamma(t_B - t_A) \\ &= \gamma\Delta t\end{aligned}$$

11.2 Length contraction

Another remarquable consequence of Lorentz transformations regards the lengths of a given object, say a ruler, measured in different inertial frames.

Let's fix the ruler to be at rest in frame O, and parallel to the x axis. The observer will be in frame O' (with velocity v).

- If $v = 0$, so that $O' = O$ and the observer is at rest relative to the ruler. Length is defined by $\Delta x = x_R - x_L = x_R$. In other words is defined to be the difference in positions of the end points of the ruler at the same time.
- If $v \neq 0$, the length is again defined by a photo taken by the observer in O'.

$$\begin{aligned}L' &= x'_R(t') - x'_L(t') \\ &= x'_R(0) - x'_L(0)\end{aligned}$$

What is the relationship between L' and L?
- Event B : $x_B = L$, $t_B = ?$, $x'_B = L'$, $t'_B = 0$.
- Event O : $x_O = t_O = x'_O = t'_O = 0$.
- Event A : $x_A = L$, $t_A = 0$, $x'_A = ?$, $t'_A = ?$.

To find the relationship, we have three methods:

1. Inverse Lorentz transformations gives

$$\begin{aligned}x_B &= \gamma x'_B \\ L &= \gamma L'\end{aligned}$$

i.e. the moving observer measures the ruler to have a length L' smaller than L.

2. The x' axis has equation $t/x = v$ and hence $t_B = vx_A = vL$. Using the Lorentz transformations applied to B we find the relationship.
3. Using the invariant interval between O and B.

Remarque : If the ruler is moving and the observer is stationary, does he see a ruler with a longer length? We find actually the same relation.

11.3 Doppler effect

Time dilatation plays an important role in many relativistic phenomena, including the Doppler effect. For example, the universe is expanding, with the velocity v of galaxies a distance at a distance r away from us with $v \propto r$ (Hubble's law, 1929). This measurement is based on the Doppler effect.

Consider a source emitting light with wavelength λ (and frequency ν). If the source moves relative to the observer, then the observer measures light with wavelength λ' (frequency ν').

In non-relativistic case, then $t' = t$ (no time dilatation…), and according to the observer the wave fronts are separated by $\lambda + v\Delta t = c\Delta t$, so that

$$\Delta t = \frac{\lambda}{c - v} \tag{11.6}$$

from which

$$\nu' = \nu \left(1 - \frac{v}{c}\right) \tag{11.7}$$

$$\lambda' = \frac{\lambda}{1 - v/c} \tag{11.8}$$

In the relativistic case $t' \neq t$. The Doppler effect formule is modified because of time dilatation and in fact Δt must be replace with $\gamma\Delta t$. The relativistic Doppler effect is

$$\nu' = \gamma\nu \left(1 - \frac{v}{c}\right) \tag{11.9}$$

$$\lambda' = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}} \tag{11.10}$$

11.4 Composition of velocities

Recall that from the Galilean transformations, the velocities of a particle viewed in two different frames was $\boldsymbol{w}' = \boldsymbol{w} - \boldsymbol{v}$. This can not be righr since it does not leave the speed of light invariant. We must use the Lorentz transformations. Consider two frames O and O' with O' moving along the x axis with velocity v.

In frame O a particle moves with velocity w. What is its velocity w' in frame O'?

$$\begin{aligned} w' &= \frac{\mathrm{d}x'}{\mathrm{d}t'} \\ &= \frac{\gamma(\mathrm{d}x - v\mathrm{d}t)}{\gamma(\mathrm{d}t - v\mathrm{d}x)} \\ &= \frac{(\mathrm{d}x/\mathrm{d}t - v)}{1 - v\mathrm{d}x/\mathrm{d}t} \\ &= \frac{w - v}{1 - vw} \end{aligned}$$

If $w = 1$ we find $w' = 1$.

Chapter 12

Minkowski spacetime, 4-vectors and Lorentz invariants

We have studied the components of (t, x, y, z). We have shown how it transforms under Lorentz transformations. We have also constructed the invariant interval Δs^2. Many other quantities transform in a similar way under Lorentz boost. We will see, for example, that (E, p_x, p_y, p_z) – where E is the energy of the particle and $\boldsymbol{p}$ its momemtum — transform in the same way:

$$E' = \gamma(E - vp_x)$$
$$p'_x = \gamma(p_x - vE)$$

and the invariant is

$$m^2 = E^2 - \boldsymbol{p}^2$$

In the following, we will construct general 4-vectors, that is vectors in space-time which transform in the same way as (t, x, y, z) under a Lorentz transformations.

12.1 4-vector

4-vectors will be denoted by $\underline{A}$. We will introduce a four dimension (Minkowski) space-time and define a scalar product between 4-vectors $(\underline{A} \cdot \underline{B})$.

Remarque : All of this will apply to general relativity.

We will start with something much simpler than 4-dimension space-time: the plane in 2D. With this example we can understand the difference between x^1 and x_1, and also introduce the concept of metric.

Let $B = (\boldsymbol{e_1}, \boldsymbol{e_2})$ $(\|\boldsymbol{e_1}\| = \|\boldsymbol{e_2}\| = 1)$ be a base. There are two ways in which to express a vector $\boldsymbol{a}$ in this basis:

$$\boldsymbol{a} = a^1\boldsymbol{e_1} + a^2\boldsymbol{e_2}$$
$$\boldsymbol{a} \cdot \boldsymbol{e_1} = a_1 \qquad \boldsymbol{a} \cdot \boldsymbol{e_2} = a_2$$

1. If B is orthonormal, $a^1 = a_1$, since

$$a_1 = \boldsymbol{a} \cdot \boldsymbol{e_1} = (a^1 \boldsymbol{e_1} + a_2 \boldsymbol{e_2})\boldsymbol{e_1} = a^1$$

2. If B is non-orthonormal, we have

$$a_1 = \boldsymbol{a} \cdot \boldsymbol{e_1} = a^1 + a^2 \boldsymbol{e_1} \cdot \boldsymbol{e_2}$$

A 4-vector $\underline{A}$ in space-time with similarly be described either by:

1. its contravariant components A^μ;
2. its covariant components A_μ.

In general $A^\mu \neq A_\mu$.

Back in the 2D Euclidean plane, consider the scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a^1 \boldsymbol{e_1} + a^2 \boldsymbol{e_2})\boldsymbol{b} = a^1 b_1 + a^2 b_2$$

From now we will use a modified summation convention (Einstein summation convention): we only sum over two repeated indices if one of them is "upstairs" and the other is "downstairs". Hence

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^1 b_1 + a^2 b_2 = a^i b_i \tag{12.1}$$

The same will apply with 4-vectors.

Reconsider the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$. Since $\boldsymbol{a} = a^i \boldsymbol{e_i}$ and $\boldsymbol{b} = b^i \boldsymbol{e_i}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^j (\boldsymbol{e_i} \cdot \boldsymbol{e_j})$.

Using this expression, we define the metric

$$g_{ij} = \boldsymbol{e_i} \cdot \boldsymbol{e_j} \tag{12.2}$$

and

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^j g_{ij} \tag{12.3}$$

and we deduce that

$$b_i = g_{ij} b^j \tag{12.4}$$

The metric is the object which enables one to raise and lower indices.

Rappel: Using the example of the 2D Euclidean plane, we discussed the difference between a_i and a^i.

– One goes between a^i and a_i using using the metric g_{ij}

$$a_i = g_{ij} a^j$$

– Modified summation convention (used all the time here after): sum only on repated indices when one is "downstairs" and the other "upstairs". Hence

$$a_i = g_{ij} a^j = \sum_{j=1}^{2} g_{ij} a^j = g_{ik} a^k$$

– Scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^j g_{ij} = a^i b_i$$

– The distance between $\boldsymbol{x}$ and $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$ is

$$\mathrm{d}\ell^2 = \mathrm{d}\boldsymbol{x}^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{12.5}$$

– By definition the metric is $g_{ij} = \boldsymbol{e_i} \cdot \boldsymbol{e_j}$ so

$$g_{ij} = g_{ji} \tag{12.6}$$

If $\boldsymbol{e_i}$ are orthogonal and normalised then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The distance $\mathrm{d}\ell^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ is invariant under rotations, translations and reflections.

For example, a rotation might be written as

$$x^i = R^i{}_j x^j$$

We wan also write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If $\boldsymbol{e_i}$ are non-orthogonal, g_{ij} change and $a^i \neq a_i$.

The metric tells us about the intrinsec geometry of a surface.

Back to special relativity.

– Here we deal with vectors $\underline{A}$ in 4-dimensional space-time.
– $\underline{A}$ can be represented either by its contravariant or covariant components.
– Greek indices take values 0 to 3. That is they run over the 4-space-time directions.
– Latin indices take values 1 to 3. That is they run over the 3 spatial directions. The time component of a vector is the $\mu = 0$ component.
– The metric of special relativity (Minkowski metric) is conventionnally denoted by $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$ (which is reserved to general relativity).

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{12.7}$$

– $\eta_{\mu\nu}$ is constant in special relativity does not change from a frame to another.
– As before we raise and lower indices using the metric

$$A_\mu = \eta_{\mu\nu} A^\nu \tag{12.8}$$

For example $A_0 = A^0$ but $A_i = -A^i$.

– The inverse metric $\eta^{\mu\nu}$ is the inverse matrix of $\eta_{\mu\nu}$.
$\underline{\underline{M}}$ with component M_{ij} and $\underline{\underline{M}}^{-1}$ with components M^{ij}. We have $\underline{\underline{M}}\underline{\underline{M}}^{-1} = I$ and $M_{ik}M^{kj} = \delta_i{}^j$. So $\eta^{\mu\nu} = \eta_{\mu\nu}$.
The components (numerical value) $\eta^{\mu\nu}$ are the same as those of $\eta_{\mu\nu}$, but in order to use the summation convention we must differentiate between the two

$$\eta^{\mu\alpha} A_\mu = \underbrace{\eta^{\mu\alpha}\eta_{\mu\nu}}_{=\delta^\alpha{}_\mu} A^\nu$$

$\eta^{\mu\alpha} = \delta^{\alpha}{}_{\nu} A^{\nu} = A^{\alpha}$ so $A^{\alpha} = \eta^{\mu\alpha} A_{\mu}$. We can go the inverse matrix to go from covariant to contravariant indices:

$$A_{\mu} = \eta_{\mu\nu} A^{\nu} \Longleftrightarrow A^{\alpha} = \eta^{\alpha\beta} A_{\beta}$$

Rules for indices: An expression such as $A^{\alpha} = \eta^{\alpha\beta} A_{\beta}$ contains two types of indices:
– α indices which are free indices (not summed over).
– β indices which are dummy indices (which are summed over).
The final expression A^{α} does not depend on β since we sum over β. The free indices must appear in the same position (that is upstairs or downstairs) on each side of the equation.

Scalar product, as before

$$\underline{A} \cdot \underline{B} = \eta_{\mu\nu} A^{\mu} B^{\nu} = A^{\mu} B_{\mu}$$

Hence $B^{i} = -B_{i}$ and

$$\underline{A}^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$

It reminds the invariant Δs and it is not positive definite.

As before, in the context of special relativity (where we discuss events, invariants...), we define $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and $(x_0, x_1, x_2, x_3) = (t, -x, -y, -z)$, hence

$$\underline{x}^2 = \underbrace{t^2 - \boldsymbol{x}^2}_{\text{invariant interval}}$$

Lorentz transformations:

$$(x^{\mu})' = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{12.9}$$

Since the interval is invariant under Lorentz transformations, $x'^2 = x^2$ or

$$\eta_{\mu\nu} x'^{\mu} x'^{\nu} = \eta_{\alpha\beta} x^{\alpha} x^{\beta}$$

but there is a problem for repeated indices so

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} x^{\alpha} \Lambda^{\nu}{}_{\beta} x^{\beta} = \eta_{\alpha\beta} x^{\alpha} x^{\beta}$$

hence

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta} \tag{12.10}$$

This identity defines the Poincaré group.

Définition 12.1 (4-vector)**.** A vector $\underline{A}$ is said to be a 4-vector if its contravariant components A^{μ} transform as $(A^{\mu})' = \Lambda^{\mu}{}_{\nu} A^{\nu}$ when $(x^{\mu})' = \Lambda^{\mu}{}_{\nu} x^{\nu}$.

Hence, by construction, $\underline{A}^2$ is invariant under Lorentz transformations. $\underline{A} \cdot \underline{B}$ is also invariant under Lorentz transformations. $(\underline{A} + \underline{B})^2$ is also invariant.

Just as the invariant interval x^2, a 4-vector $\underline{A}$ is classified according to the sign of $\underline{A}^2$:
– if $\underline{A}^2 = 0$ then $\underline{A}$ is light-like;
– if $\underline{A}^2 > 0$ then $\underline{A}$ is time-like;
– if $\underline{A}^2 < 0$ then $\underline{A}$ is space-like.

4-vectors can have some very strange properties: for example, consider two light-like 4-vectors $\underline{A}$ and $\underline{B}$. Suppose that $\underline{B} \parallel \underline{A}$, i.e. $\underline{B} = \lambda \underline{A}$ ($\lambda \in \mathbb{R}$). But $\underline{B}$ is also perpendicular to $\underline{A}$:

$$\underline{A} \cdot \underline{B} = \underline{A} \cdot \lambda \underline{A} = \lambda \underline{A}^2 = 0$$

Consider again two general 4-vectors $\underline{A}$ and $\underline{B}$, then $\underline{AB}$ is also a 4-vector, since

$$(A^\mu + B^\mu)' = A'^\mu + B'^\mu = \Lambda^\mu{}_\nu A^\nu + \Lambda^\mu{}_\nu B^\nu = \Lambda^\mu{}_\nu (\underline{A} + q\underline{B})$$

Hence $(\underline{A}+\underline{B})^2$ is Lorentz-invariant. But $\underbrace{(\underline{A}+\underline{B})^2}_{\text{invariant}} = \underbrace{\underline{A}^2}_{\text{invariant}} + 2\underline{A}\,\underline{B} + \underbrace{\underline{B}^2}_{\text{invariant}}$ hence $\underline{A}\,\underline{B}$ is invariant.

More generally, a quantity is Lorentz-invariant (by construction) if it contains no free indices.

Exemple 12.1.

1. $\underline{A}\,\underline{B} = A^\mu B^\nu \eta_{\mu\nu}$ contains no free indices and is Lorentz-invariant.
2. $A^\mu B_\nu$ contains the two free indices μ and ν (which are not summed over). This is not a Lorentz-invariant.
3. $C^\mu B_\nu C_\mu$ is not Lorentz-invariant.

12.2 Some 4-vectors

Some 4-vectors contravariant are:

- The position x^μ.
- The velocity $u^\mu = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}s}$ [1].
- The acceleration $a^\mu = \dfrac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2}$.
- The momemtum $p^\mu = m u^\mu$.

Other 4-vectors such as $\partial_\mu = \partial/\partial x^\mu$ are covariant.

A 4-vector $\underline{A}$ with covariant components A_μ transforms as

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu$$

under a Lorentz transformation.

What is $\Lambda_\mu{}^\nu$?

Apply the rules starting from $\Lambda^\alpha{}_\beta$.

1. Lower the index α:
$$\Lambda_{\mu\beta} = \eta_{\alpha\mu} \Lambda^\alpha{}_\beta$$
2. Raise the index β:
$$\Lambda_{\mu\beta} \eta^{\beta\nu} = \Lambda_\mu{}^\nu$$

One can also use the transformation properties of A^α:

$$A'_\mu = \eta_{\mu\nu} A'^\nu = \eta_{\mu\nu} \Lambda^\nu{}_\gamma A^\gamma = \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\gamma \eta^{\gamma\varepsilon}}_{=\Lambda_\mu{}^\varepsilon} A_\varepsilon$$

We will see later that ∂/∂_μ indeed transforms as a covariant-vector.

1. Where $\mathrm{d}s$ is the invariant interval.

Chapter 13

Different 4-vectors and their invariant

13.1 Velocity

By definition

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{13.1}$$

where τ is the proper-time defined by $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$.

u^μ are indeed the contravariant components of a 4-vector ($\underline{u}$) since τ is Lorentz invariant: hence under a Lorentz transformation,

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu$$

Notice that $\mathrm{d}x^\mu/\mathrm{d}t$ can not be the contravariant components of any 4-vector since under a Lorentz transformation

$$\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right)' = \frac{\Lambda^\mu{}_\nu}{\Lambda^0{}_\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^\alpha} \neq \Lambda^\mu{}_\gamma \frac{\mathrm{d}x^\gamma}{\mathrm{d}t}$$

So

$$\begin{aligned} u^0 &= \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ u^i &= \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}t} \end{aligned}$$

What is the invariant

$$\begin{aligned} \underline{u}^2 &= u^\mu u^\nu \eta_{\mu\nu} \\ &= (u^0)^2 - (\boldsymbol{u})^2 \\ &= c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)^2 \\ c^2 &= \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 (c^2 - \boldsymbol{v}^2) \end{aligned}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$$

Hence

$$\underline{u} = (\gamma c, \gamma \boldsymbol{v}) \tag{13.2}$$

then $c^2 = \underline{u}^2$. Thus u is a time like 4-vector.

13.1.1 Composition of velocities

Let $\boldsymbol{v}$ and $\boldsymbol{v'}$ be the velocities of particle in the original frame and the frame moving with velocity β. Since $(u^0)' = \gamma(\beta)(u^0 - \beta u^1)$ and $(u^1)' = \gamma(u^1 - \beta u^0)$ and $u^\mu = \gamma(v)(c, \boldsymbol{v})$ and $u'^\mu = \gamma(v')(c, \boldsymbol{v}')$, the equation is

$$\gamma(v') = \gamma(\beta)\Big(\gamma(v) - \beta\gamma(v)v_x\Big)$$

13.2 Acceleration

The 4-vector acceleration is defined by

$$\underline{A} = \frac{\mathrm{d}\underline{u}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\underline{x}}{\mathrm{d}\tau^2} \tag{13.3}$$

Its components are

$$A^\mu = \Big(\gamma^4 \boldsymbol{v}\cdot\boldsymbol{a}, \gamma^4(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v} + \gamma^2\boldsymbol{a}\Big) \tag{13.4}$$

Since $\underline{u}^2 = c^2$ we have

$$2\underline{u}\frac{\mathrm{d}\underline{u}}{\mathrm{d}\tau} = 0$$

so

$$\underline{u}\cdot\underline{A} = 0 \tag{13.5}$$

Thus $\underline{A}$ is perpendicular to $\underline{u}$.

Since $\underline{u}$ is time-like vector, it follows from (13.4) that $\underline{A}\cdot\underline{u}$ is a space-like vector.

13.3 Derivative $\underline{\partial}$

Consider

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{13.6}$$

Are there components of a 4-vector? If so, is it "naturally" covariant or contravariant?

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha} \tag{13.7}$$

where since $x'^\beta = \Lambda^\beta{}_\alpha x^\alpha$, it follows that $x^\alpha = \Lambda_\mu{}^\alpha x'^\mu$ (recall that $\Lambda_\varepsilon{}^\mu \Lambda^\varepsilon{}_\omega = \delta^\mu{}_\omega$), thus

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \Lambda_\mu{}^\alpha \tag{13.8}$$

and

$$\frac{\partial}{\partial x'^{\mu}} = \Lambda_{\mu}{}^{\alpha} \frac{\partial}{\partial x'^{\alpha}} \tag{13.9}$$

This is the transform rule for the covariant components of a 4-vector. Hence we define

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} \tag{13.10}$$

to be the covariant of the 4-vector $\underline{\partial}$.

Why is this useful?

Exemple 13.1.

$\underline{\partial}^2$ is Lorentz invariant and

$$\underline{\partial}^2 = \partial_0^2 - \boldsymbol{\nabla}^2$$

13.4 Wave $\underline{k}$

The wave 4-vector is defined by

$$k^{\mu} = (\omega, \boldsymbol{k}) \tag{13.11}$$

where ω is the angular frequency and $\boldsymbol{x}$ the wave vector (with $\boldsymbol{k}^2 = \omega^2$). Indeed, the phase of an electromagnetic wave

$$\varphi = \omega t - \boldsymbol{k} \cdot \boldsymbol{x} = \underline{x} \cdot \underline{k} \tag{13.12}$$

is a Lorentz scalar. $\underline{k}$ is a light-like 4-vector since $\underline{k}^2 = 0$.

13.4.1 Application

Consider a source emitting light with angular frequency ω at an angle θ to the x-axis in a frame O.

According to an observer in frame O' moving with velocity v along the x-axis, the angular frequency is ω' and the source is at angle θ' with the x'-axis.

The different components of k^{μ} in the two frames are

$$\begin{aligned} k^{\mu} &= (\omega, -\omega\cos\theta, -\omega\sin\theta) \\ k'^{\mu} &= (\omega', -\omega'\cos\theta', -\omega'\sin\theta') \end{aligned}$$

since $|\boldsymbol{k}| = \omega$. But k^{μ} are the components of a 4-vector then

$$\begin{aligned} \omega' &= \gamma(\omega - vk) \\ &= \gamma\omega(1 + \cos\theta) \end{aligned}$$

(relativistic Doppler effect).

13.5 Momentum

The momentum 4-vector is defined by

$$\underline{P} = m\underline{u} \tag{13.13}$$

where m must be a Lorentz invariant. Hence m is an invariant characteristic of a particle: its mass.

Remarque : Note that $\underline{P}$ is not defined for massless particles propagating at the speed of light . For photons, we use $\underline{k}$.

The invariant

$$\underline{P}^2 = m^2\underline{u} = m^2c^2 \tag{13.14}$$

and the components of $\underline{P}$ are

$$P^\mu = (\gamma mc, \gamma m\boldsymbol{v}) \tag{13.15}$$

What is P^0? Consider the non-relativistic limit, then

$$\begin{aligned} P^0 &\approx mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) \\ &\approx \frac{1}{c}\left(mc^2 + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \cdots\right) \end{aligned}$$

This motivates the definition, valid for all $\boldsymbol{v}$ of

$$P^0 = \frac{E}{c} \tag{13.16}$$

where E is the total energy of the particle.

The invariant is

$$E^2 = \boldsymbol{p}^2c^2 + m^2c^4 \tag{13.17}$$

because

$$\underline{P}^2 = m^2c^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2$$

If the particle is at rest, $\boldsymbol{p} = 0$ and $E = mc^2$, the rest mass of a particle.

For a massless particle $E = |p|$.

13.6 Force

We must write down a generalisation of Newton law, namely an equation of the form "$f = ma$", but which transform correctly under Lorentz transformation. Hence it must be in terms of 4-vectors

$$\underline{F} = m\underline{A} \tag{13.18}$$

and hence

$$\underline{F} \cdot \underline{u} = 0 \tag{13.19}$$

The obvious question is: what is the link between the components F^μ and the usual 3-force $\boldsymbol{f}$?

Recall (13.4). Does there exist a frame in which $\boldsymbol{A}$ reduces to $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$? If so, in that frame $\boldsymbol{F} = \boldsymbol{f}$. The answer is yes: when $\boldsymbol{v} = 0$ so that $\gamma = 1$ and $A^\mu_{IRF} = (0, \boldsymbol{a})$ (IRF means instantaneous rest frame). In this frame $F^\mu_{IRF} = (0, \boldsymbol{f})$ and $u^\mu_{IRF} = (1, 0)$.

Since F^μ is a 4-vector, we can now find its components in any frame using the inverse Lorentz transformations:

$$F^\mu = \gamma(\boldsymbol{v} \cdot \boldsymbol{f}, \boldsymbol{f}) \tag{13.20}$$

and we have

$$F^\mu = \frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} \tag{13.21}$$

$\boldsymbol{v} \cdot \boldsymbol{f}$ is the work done by the force.

Chapter 14

Particle collisions

In classical mechanics, particle collisions are straight forward to study if the particle are free (i.e. no interactions between them):

$$p_1 + p_2 + \cdots \longrightarrow q_1 + q_2 + \cdots$$

We use the conservations of energy and momemtum:

$$\sum_{ini} E_i = \sum_{fin} E_f$$
$$\sum_{ini} \boldsymbol{p_i} = \sum_{fin} \boldsymbol{p_f}$$

In special relativity, the unique equation is

$$\sum_{ini} \underline{P}_i = \sum_{fin} \underline{P}_f \tag{14.1}$$

since $P^\mu = (E, \boldsymbol{p})$.

Furthermore, since each particle also has a rest-mass energy (mc^2), regarding energy conservation, some of that mass can be converted into energy and vice versa. Hence the initial number of particles is not necessarly equal to the final number of particles:

$$p_1 + \cdots + p_r \longrightarrow q_1 + \cdots + q_s$$

Generally $r \neq s$ and

$$\sum_{i=1}^{r} \underline{P}_i = \sum_{f=1}^{s} \underline{P}_f$$

A collision is said to be elastic if the initial and final particles are the same:

$$a + b \longrightarrow a + b$$
$$e^- + \gamma \longrightarrow e^- + \gamma \qquad \text{(Compton effect)}$$

else, it's said to be inelastic:

$$e^+ + e^- \longrightarrow \gamma \qquad \text{(annihilation of particles)}$$
$$\pi^0 \longrightarrow \gamma + \gamma \qquad \text{(uncharged pion)}$$
$$n \longrightarrow p + e^- + \overline{\nu} \qquad (\beta\text{-decay/radioactivity})$$
$$p + p \longrightarrow p + p + p + \overline{p} \qquad \text{(proton-proton scattering)}$$

The analysis of such collisions can be simplified by recalling that:
- For each collision you can construct many invariants, for example $\underline{p}_1$, $(\underline{p}_1 + \underline{p}_2)^2 \dots$
- One can easily pass from one reference frame to another using Lorentz transformations. Indeed in certain reference frames, calculations can be much simpler.

Typically three frames come into these types of calculations:
- The laboratory frame: the one in which the experiment is done and hence the one in which you must present your results!
- Center of mass frame: in which the total spatial momentum vanishes:

$$\sum_{i=1}^{r} \boldsymbol{p_i} = \sum_{f=1}^{s} \boldsymbol{p_f}$$

- Frame attached to one of the particles.

14.1 Examples

14.1.1 An elastic collusion: the Compton effect

$$e^- + \gamma \longrightarrow e^- + \gamma$$

One use also the name "Compton scattering".

Take as laboratory frame the frame in which the initial electron is at rest. The photon and the electron are scattered with an angle θ and φ. Find the wavelength λ if the scattered photon in terms of the wavelength λ of the initial photon, the electron mass m and the photon diffusion angle θ.

Initial

- $P^\mu_{e^-} = (m, \boldsymbol{0})$
- $P^\mu_\gamma = (|\boldsymbol{p_\gamma}|, \boldsymbol{p_\gamma})$
- $|\boldsymbol{p_\gamma}| = \hbar\omega = h/\lambda$

Final

- $\hat{P}^\mu_{e^-} = (E, \boldsymbol{p})$ with $E^2 = m^2 + |\boldsymbol{p}|^2$
- $\hat{P}^\mu_\gamma = (|\boldsymbol{p_\gamma}'|, \boldsymbol{p_\gamma}')$
- $|\boldsymbol{p_\gamma}'| = h/\lambda'$

Conservation of 4-momentum:

$$\underline{P}_{e^-} + \underline{P}_\gamma = \underline{\hat{P}}_{e^-} + \underline{\hat{P}}_\gamma$$

It's not a good idea to square the expression, because we would have φ.

$$\underline{P}_{e^-} + \underline{P}_\gamma - \underline{\hat{P}}_\gamma = \underline{\hat{P}}_{e^-}$$
$$(\underline{P}_{e^-} + \underline{P}_\gamma - \underline{\hat{P}}_\gamma)^2 = \underline{\hat{P}}_{e^-}^2 = m^2$$
$$\implies \underline{P}_{e^-} + \underbrace{\underline{P}_\gamma}_{=0} - \underbrace{\underline{\hat{P}}_\gamma}_{=0} = m$$
$$\implies 2(\underline{P}_{e^-} \cdot \underline{P}_\gamma - \underline{P}_{e^-} \cdot \underline{\hat{P}}_\gamma - \underline{P}_\gamma \cdot \underline{\hat{P}}_\gamma) + m^2 = m^2$$
$$\implies \underline{P}_{e^-} \cdot \underline{P}_\gamma = \underline{P}_{e^-} \cdot \underline{\hat{P}}_\gamma + \underline{P}_\gamma \cdot \underline{\hat{P}}_\gamma$$
$$\implies m\frac{h}{\lambda} = m\frac{h}{\lambda'} + \left(\frac{h}{\lambda'}\frac{h}{\lambda} - \boldsymbol{p_\gamma} \cdot \boldsymbol{p_\gamma}'\right)$$

where $\boldsymbol{p_\gamma} \cdot \boldsymbol{p_\gamma}' = |\boldsymbol{p_\gamma}|\ |\boldsymbol{p_\gamma}'| \cos\theta$, and since $|\boldsymbol{p_\gamma}| = h/\lambda$

$$\boxed{\lambda' - \lambda = \frac{h}{m}(1 - \cos\theta)} \tag{14.2}$$

14.1.2 Elastic proton scattering

$$p + p \longrightarrow p + p$$

- Suppose that in the experiment (laboratory frame) which studies this collision, one of the initial protons is at rest. Let E be the energy of moving proton.
- What is the energy E' of each of the initial protons in the center of mass frame? (see table 14.1)

	Laboratory	Center of mass
Proton 1	$p_1^\mu = (m, \boldsymbol{0})$	$p_1'^\mu = (E', \boldsymbol{p})$
Proton 2	$p_2^\mu = (E, \boldsymbol{q})$	$p_2'^\mu = (E', -\boldsymbol{p})$

Table 14.1: Impulsion 4-vector in the laboratory and center of mass frames

We have

$$(\underline{P}_1 + \underline{P}_2)^2 = (\underline{P}'_1 + \underline{P}'_2)^2$$
$$\underline{P}_1^2 + \underline{P}_2^2 + 2\underline{P}_1 \cdot \underline{P}_2 = (2E')^2$$
$$m^2 + m^2 + 2(Em) = 4E'^2$$
$$m^2 + Em = 2E'^2$$

so

$$E' = \sqrt{\frac{1}{2}m(E + m)} \tag{14.3}$$

14.1.3 Desintegration

$$a \longrightarrow b + c$$

a, b, and c are some particles with masses m_a, m_b and m_c.

1. Given m_b and m_c, can this desintegration occur for all values of m_a?
2. In the center of mass frame, what is the 3-momentum of b and c as a function of m_a, m_b and m_c?

To answer to the first question, we must work in the center of mass frame, since particles b and c have minimum energy when their 3-momentum vanishes ($\boldsymbol{p_b} = \boldsymbol{p_c} = \mathbf{0}$). But this is the definition since $\boldsymbol{p_b} + \boldsymbol{p_c} = \mathbf{0}$!

In the center of mass frame:

$$P_a'^{\mu} = (m_a, \mathbf{0})$$
$$P_b'^{\mu} = (E'^b, \boldsymbol{p}) = (\sqrt{m_b^2 + \boldsymbol{p}^2}, \boldsymbol{p})$$
$$P_c'^{\mu} = (E'^c, -\boldsymbol{p}) = (\sqrt{m_c^2 + \boldsymbol{p}^2}, -\boldsymbol{p})$$

Conservation of $\underline{P}$: from the $\mu = 0$ component of

$$P_a'^{\mu} = P_b'^{\mu} + P_c'^{\mu}$$
$$m_a = \sqrt{m_b^2 + \boldsymbol{p}^2} + \sqrt{m_c^2 + \boldsymbol{p}^2}$$
$$m_a \geq m_b + m_c$$

since $\boldsymbol{p}^2 > 0$.

What is $|\boldsymbol{p}|$?

$$(E'^b)^2 = m_b^2 + \boldsymbol{p}^2$$
$$(E'^c)^2 = m_c^2 + \boldsymbol{p}^2$$
$$\Longrightarrow (E'^b)^2 - m_b^2 = (E'^c)^2 - m_c^2$$

and we also have

$$m_a = E_b' + E_c'$$

so substitute $E_b' = m_a - E_c'$ and this gives

$$(m_a - E_c')^2 - m_b^2 = (E_c')^2 - m_c^2$$
$$\Longrightarrow m_a^2 - 2E_c' m_a - m_b^2 = -m_c^2 \Longrightarrow E_c^* = \frac{m_a^2 - m_b^2 + m_c^2}{2m_a}$$

Finally since $(E_c')^2 = m_c^2 + \boldsymbol{p}^2$:

$$|\boldsymbol{p}|^2 = (E_c')^2 - m_c^2 = \left(\frac{m_a^2 - m_b^2 + m_c^2}{2m_a}\right)^2 - m_c^2$$

14.1.4 Inelastic collision of protons

$$p + p \longrightarrow p + p + p + \overline{p}$$

In the laboratory frame, one of the protons is at rest. What is the minimum of energy E_{min} that the other proton must have in order for process the reaction to occur?

It's easiest to calculate the minimum energy in the center of mass frame, and then to transform to the laboratory frame. The conservation of energy in center of mass frame is

$$2E'_{min} = 4m$$

Now need to transform to the laboratory frame in order to determine E_{min}. Again use the invariant:

$$\begin{aligned}
(\underline{P}_1 + \underline{P}_2)^2 &= (\underline{P}'_1 + \underline{P}'_2)^2 \\
\underline{P}_1^2 + \underline{P}_2^2 + 2\underline{P}_1 \cdot \underline{P}_2 &= \underline{P}_1'^2 + \underline{P}_2'^2 + 2\underline{P}'_1 \cdot \underline{P}'_2 \\
m^2 + m^2 + 2mE_{min} &= m^2 + m^2 + 2\Big((E'_{min})^2 + \boldsymbol{q}^2\Big) \\
mE_{min} &= (E'_{min})^2 + \boldsymbol{q}^2
\end{aligned}$$

thus

$$\begin{aligned}
mE_{min} &= (E'_{min})^2 + \Big((E'_{min})^2 - m^2\Big) \\
&= 4m^2 + (4m^2 - m^2) \\
&= 7m^2
\end{aligned}$$

so $E_{min} = 7m$. But $(E'_{min})^2 = m^2 + \boldsymbol{q}^2$.

Chapter 15

Electromagnetism

Our starting point was Maxwells equations: c is a Lorentz invariant and the wave equations are invariant under Lorentz transformations.

Now we want to try to rewrite Maxwells equations themselves in the language of 4-vectors.

We are used to solve Maxwells equations in simple situation (eg: a charge Q at rest). But what if the charge is moving? But to go from a stationary to a moving charge we only need to do a Lorent transformation: if we know $\boldsymbol{E}$ and $\boldsymbol{B}$ transform under a Lorentz transformation, we could easily find $\boldsymbol{E}'$ and $\boldsymbol{B}'$.

Can one construct some 4-vectors from $\boldsymbol{E}$ and $\boldsymbol{B}$? The answer is no: but they can be unified into a 4-tensor $F_{\mu\nu}$, which we will construct.

15.1 Tensors

A number of physical quantities are neither scalars (Lorentz invariants), nor the components of 4-vectors, but the components of 4-tensors.

A tensor generally has $n \geq 2$ indices which can be covariant or contravariant. An example is $\eta^{\mu\nu}$, $\eta_{\mu\nu}$, $\eta^{\mu}{}_{\nu}$. A general tensor is typically denoted by the letter T (for tensor).

Exemple 15.1.
For $n = 4$: $T^{\mu\nu\alpha\beta}$, $T^{\mu}{}_{\gamma}{}^{\delta}{}_{\varepsilon} \cdots$

Indices on a tensor are raised and lower using the same rules as a 4-vector, but with the additional rule that order of indice on the tensor is left unchanged.

Exemple 15.2.
Since $u^{\alpha} = \eta^{\alpha\beta} u_{\beta}$

$$T^{\alpha}_{\gamma} = \eta^{\alpha\beta}_{\beta\gamma}$$

Under a Lorentz transformation, an $n = 2$ transforms in the same way as two 4-vectors with the same index structures. So since

$$A^{\alpha} B^{\beta} \longrightarrow \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} (A^{\gamma} B^{\delta})$$

then

$$T^{\alpha\beta} \longrightarrow T'^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} T^{\gamma\delta} \tag{15.1}$$

$$T^{\alpha}{}_{\beta} \longrightarrow T'^{\alpha}_{\beta} = \Lambda^{\alpha}{}_{\gamma} \Lambda_{\delta}{}^{\beta} T^{\gamma}{}_{\delta} \tag{15.2}$$

15.2 Electromagnetic 4-tensor

$\boldsymbol{E}$ and $\boldsymbol{B}$ will be the components of "naturally" contravariant 4-tensor with two indices $F^{\mu\nu}$, the electromagnetic 4-tensor.

To construct $F^{\mu\nu}$, we need to go back to Maxwell's equations

$$\nabla\,\boldsymbol{E} = \rho \tag{15.3a}$$

$$\nabla\times\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{15.3b}$$

$$\nabla\,\boldsymbol{B} = 0 \tag{15.3c}$$

$$\nabla\times\,\boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{15.3d}$$

From (15.3a) and (15.3d) we first derive the continuity equation

$$\frac{\partial\rho}{\partial t} = \nabla\,\frac{\partial \boldsymbol{E}}{\partial t} = \nabla(\nabla\times\,\boldsymbol{B} - \boldsymbol{j}) = -\,\nabla\,\boldsymbol{j}$$

$$\Longrightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\,\boldsymbol{j} = 0} \tag{15.4}$$

Time in all inertial frames, i.e if must be possible to write this equation in explicitly Lorentz invariant form:

$$\partial_\mu j^\mu = 0 \tag{15.5}$$

where j^μ is the current 4-vector with components

$$j^\mu = (\rho, \boldsymbol{j}) \tag{15.6}$$

Now consider the two Maxwell equations (15.3c) and (15.3b), and as usual introduce the potentials ϕ and $\boldsymbol{A}$.

$$\boldsymbol{B} = \nabla\times\,\boldsymbol{A} \tag{15.7a}$$

$$\boldsymbol{E} = -\,\nabla\,\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{15.7b}$$

However it's important to note that given $\boldsymbol{E}$ and $\boldsymbol{B}$, there is no unique way of constructing ϕ and $\boldsymbol{A}$: the transformations

$$\boldsymbol{A}' = \boldsymbol{A} - \nabla\,f \tag{15.8a}$$

$$\phi' = \phi + \frac{\partial f}{\partial t} \tag{15.8b}$$

leave $\boldsymbol{E}$ and $\boldsymbol{B}$ invariant.

It's useful to restrict the possible choices for ϕ and $\boldsymbol{A}$ by imposing some extra-conditions: a particularly useful choice is the so-called Lorenz gauge:

$$\frac{\partial\phi}{\partial t} + \nabla\,\boldsymbol{A} = 0 \tag{15.9}$$

In terms of ϕ and $\boldsymbol{A}$, Maxwell equations with sources (15.3a) and (15.3d) become (using (15.9) and (15.7))

$$(\partial_t^2 - \nabla^2)\phi = \rho$$

$$(\partial_\mu\partial^\mu)\phi = j^0$$

and

$$(\partial_t^2 - \nabla^2)\boldsymbol{A} = \boldsymbol{j}$$
$$(\partial_\mu \partial^\mu)\boldsymbol{A} = \boldsymbol{j}$$

Since $(j^0, \boldsymbol{j})$ are the components of a 4-vector, and since $\partial_\mu \partial^\mu$ is a Lorentz invariant, $(\phi, \boldsymbol{A})$ are the components of a 4-vector

$$A^\mu = (\phi, \boldsymbol{A}) \tag{15.10}$$

and the previous equation become

$$(\partial_\mu \partial^\mu) A^\nu = j^\nu \tag{15.11}$$

From (15.7), the derivatives of A^μ give the electric and magnetic fields, hence define

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} \tag{15.12}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{15.13}$$

For example

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 \\ &= \partial^0 A^1 - \partial^1 \phi \\ &= \frac{\partial A^1}{\partial t} + \frac{\partial \phi}{\partial x} \end{aligned}$$

Note that $F^{\mu\nu} = -F^{\nu\mu}$, i.e $F^{\mu\nu}$ is an antisymmetric tensor.

In terms of $F^{\mu\nu}$, Maxwells equations (15.3a) and (15.3d) can be written as

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{15.14}$$

and Maxwells equations without sources (15.3c) and (15.3b) are

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{15.15}$$

Finally one can construct from $F_{\mu\nu}$ a number of different Lorentz invariant. For example

$$F_{\mu\nu} F^{\mu\nu} \propto \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{15.16}$$

which is, in fact, the Lagrangian of electromagnetism.

15.3 Transformations of E and B

By definition

$$F^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$$

so for example take $\mu = 0$ and $\nu = 1$.

$$\Lambda^0{}_0 = \gamma$$
$$\Lambda^0{}_1 = -v\gamma$$
$$\Lambda^1{}_0 = -v\gamma$$
$$\Lambda^1{}_1 = \gamma$$

so

$$\begin{aligned} F'^{01} = -E'_x &= \Lambda^0{}_0\Lambda^1{}_\beta F^{0\beta} + \Lambda^0{}_1\Lambda^1{}_\beta F^{1\beta} \\ &= \Lambda^0{}_0\Lambda^1{}_1 F^{01} + \Lambda^0{}_1\Lambda^1{}_0 F^{10} \\ &= \gamma^2(-E_x) + (-v\gamma)^2 E_x \\ &= -\gamma^2 E_x(1 - v^2) \\ &= -E_x \end{aligned}$$

so

$$E'_x = E_x$$

Taking $\mu = 1$ and $\nu = 3$

$$\begin{aligned} F'^{13} = B'_y &= \Lambda^1{}_\alpha\Lambda^3{}_\beta F^{\alpha\beta} \\ &= \Lambda^1{}_0\Lambda^3{}_\beta F^{0\beta} + \Lambda^1{}_1\Lambda^3{}_\beta F^{3\beta} \\ &= \Lambda^1{}_0\Lambda^3{}_3 F^{03} + \Lambda^1{}_1\Lambda^3{}_3 F^{13} \\ &= (-v\gamma)(-E_z) + \gamma B_y \end{aligned}$$

so

$$B'_y = \gamma(B_y + vE_z)$$

Doing the same for all values of μ and ν gives

$$\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp - \boldsymbol{v} \times \boldsymbol{B}_\perp) \tag{15.17a}$$
$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \tag{15.17b}$$
$$\boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}_\perp - \boldsymbol{v} \times \boldsymbol{E}_\perp) \tag{15.17c}$$
$$\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \tag{15.17d}$$
$$\tag{15.17e}$$

i.e

$$E'_x = E_x$$
$$E'_y = \gamma(E_y - vB_z)$$
$$E'_z = \gamma(E_z - vB_y)$$

15.4 An example: moving charge

In frame O, a charge q has constant velocity v along the x-axis. At $t = 0$, the charge is at $(x, y) = (0, 0)$. Calculate the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields generated by the charge, measured by an observer at rest in O, at position $(0, -b)$.

Introduce a frame O' in which the charge is at rest at $(x', y') = (0, 0)$. In this frame

$$\boldsymbol{B}' = 0$$
$$\boldsymbol{E}' = \frac{q}{|\boldsymbol{r}'|^3}\,\boldsymbol{r}'$$

where $\boldsymbol{r}' = (x', y')$.

At the position of the observer

$$y' = -b$$
$$x' = \gamma(x - vt) = -\gamma vt$$

so

$$\boldsymbol{E}' = \frac{q}{(x'^2 + y'^2)^{3/2}} = \frac{q}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \times (-\gamma vt, -b)$$

hence

$$E'_x = \frac{-\gamma vtq}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}$$

and

$$E'_y = \frac{-bq}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}$$

Chapter 16

Relativistic lagrangian

Clearly $L = T - V$ can not work for relativistic systems, since we know that for relativistic systems Newton equation is wrong and should be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{\boldsymbol{x}}) = 0$$

– How do we solve this problem?
– For relativistic systems we still use an action principle, but some things must change in order to be consistent with the two postulates of special relativity.

1. Time t has been treated as a special coordinates, distinct from (x, y, z). This is contrary to special relativity in which we have seen that space and time must be placed on an equal footing. So we must replace t in the definition of S by a parameter which is Lorentz invariant, and which traces the position of the particle in configuration space. A good candidate is τ, the proper time of the particle.
2. Physics must be the same in all inertial frame (so the Euler–Lagrange equations) and so L must be a Lorentz invariant, and hence also S.

Back to the free massive particle: let's choose it's proper time τ to label the position of the particle in configuration space.

$$S = \int L\mathrm{d}\tau$$

L must have dimension of energy: the simplest Lorentz invariant we could write down for a particle od mass m is

$$L = -mc^2 \tag{16.1}$$

Let's see what it gives!

$$\begin{aligned} S &= -\int mc^2\mathrm{d}\tau \\ &= \int \frac{mc^2}{\gamma}\,\mathrm{d}t \end{aligned}$$

and

$$L = -\frac{mc^2}{\gamma} \tag{16.2}$$

Notice that in non-relativistic limit

$$S = \int \mathrm{d}t \left(-mc^2 + \frac{1}{2} m \boldsymbol{x}^2 + \cdots \right) \tag{16.3}$$

The Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \frac{\partial L}{\partial \boldsymbol{x}}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} (\gamma m \dot{\boldsymbol{x}}) = 0$$

Part III

Exercices

Chapter 17

TD 1

17.1 Rappels

Exercice 17.1. Soit $f = f(z(x), y(x), x)$. Exprimer la dérivée totale $\mathrm{d}f/\mathrm{d}x$ en fonction des différentes dérivées partielles de f.

Soit $f = f(z(x), y(x), x)$. Alors

$$\begin{aligned} f(x+\delta x) &= f(z(x+\delta x), y(x+\delta x), x+\delta x) \\ &= f\left(z(x) + \frac{\mathrm{d}z}{\mathrm{d}x}\delta x, y(x) + \frac{\mathrm{d}y}{\mathrm{d}x}\delta x, x+\delta x\right) \\ &= f(z(x), y(x), x) + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}\delta x + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\delta x + \frac{\partial f}{\partial x}\delta x \end{aligned}$$

d'où

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial x}$$

Exercice 17.2. Calculer $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial \dot{x}}$, $\frac{\partial f}{\partial t}$ et $\frac{\mathrm{d}f}{\mathrm{d}t}$ (où $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $x = x(t)$ et $y = y(t)$) pour

1. $f = x^2$
2. $f = y \ln x$ avec $x(t) = t^3$
3. $f = \mathrm{sh}(\dot{x}^4 + x) + t$ avec $x(t) = \ln t$
4. $f = t^2 x y^3$ avec $x(t) = t^2$ et $y(t) = 1/t$

1. Soit $f = x^2$, alors

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial t} = 0$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2x\dot{x}$$

2. Soit $f = y \ln x$ avec $x(t) = t^3$, alors

$$\frac{\partial f}{\partial x} = \frac{y}{x}$$
$$\frac{\partial f}{\partial y} = \ln x$$
$$\frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial t} = 0$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{3yt^2}{x} + \dot{y} \ln x$$

3. Soit $f = \operatorname{sh}(\dot{x}^4 + x) + t$ avec $x(t) = \ln t$, alors

$$\frac{\partial f}{\partial x} = \operatorname{ch}(\dot{x}^4 + x)$$
$$\frac{\partial f}{\partial y} = 0$$
$$\frac{\partial f}{\partial \dot{x}} = 4\dot{x}^3 \operatorname{ch}(\dot{x}^4 + x)$$
$$\frac{\partial f}{\partial t} = 1$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = (4\ddot{x}\dot{x}^3 + \dot{x}) \operatorname{ch}(\dot{x}^4 + x)$$

4. Soit $f = t^2 x y^3$ avec $x(t) = t^2$ et $y(t) = 1/t$, alors

$$\frac{\partial f}{\partial x} = t^2 y^3$$
$$\frac{\partial f}{\partial y} = 3xt^2y^2$$
$$\frac{\partial f}{\partial \dot{x}} = 0$$
$$\frac{\partial f}{\partial t} = 2xty^3$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2t^2y^3\dot{x} - 3xy^2\dot{y} + 2xty^3$$

Exercice 17.3. Soit $f = f(z(t), \dot{z}(t), t)$. Réfléchir à la différence entre $\partial_t f$ et $\mathrm{d}_t f$.

Exercice 17.4. Passage des coordonnées cartésiennes (vecteurs unitaires $\boldsymbol{i}$ et $\boldsymbol{j}$) aux coordonnées polaires (vecteurs unitaires $\boldsymbol{e_r}(t)$ et $\boldsymbol{e_\theta}(t)$).

Soit

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

la position d'une particule au temps t en coordonnées cartésiennes.

En coordonnées polaires nous écrivons

$$\boldsymbol{r}(t) = r(t)\boldsymbol{e_r}(t)$$

Exprimer $\dot{\boldsymbol{r}}(t)$ et $\ddot{\boldsymbol{r}}(t)$ dans les deux jeux de coordonnées.

17.2 Introduction aux fonctionnelles

Exercice 17.5. Sur le plan euclidien, la distance parcourue L entre deux points donnés O et A est un exemple de fonctionnelle : cette distance, $L[y]$, dépend du chemin $y(x)$ qu'on emprunte entre les points.

Soient $O(0,0)$ et $A(x_1, y_1)$, et soit $y(x)$ une courbe qui va de O à A. Utiliser votre intuition pour dire quelle courbe va minimiser $L[y]$.

1. Pour $y(x) = (y_1/x_1)x$, quelle est la valeur de cette distance minimale L_{min} ?
2. Écrire la forme de $L[y]$ pour une courbe générale $y(x)$.

1. Puisque OA est une droite, on a

$$L_{min} = \sqrt{x_1^2 + y_1^2}$$

2. On a

$$L = \int \mathrm{d}\ell$$

avec

$$\begin{aligned} \mathrm{d}\ell &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \\ &= \sqrt{\mathrm{d}x^2 + \mathrm{d}x^2 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \end{aligned}$$

d'où

$$L[y] = \int_0^{x_1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$

Exercice 17.6. Cet exercice est une introduction au fameux problème de la courbe brachistochrone.

Considérons dans le plan vertical deux points A et B. On lâche une masse, sans vitesse initiale, en A. La masse glisse sans frottements sur un guide sous l'effet de la pesanteur et arrive en B. Le problème entier du brachistochrone consiste à trouver la forme de la courbe $y(x)$ du guide telle que le temps nécessaire à la masse pour aller de A en B soit minimum. Ici nous allons faire un exemple plus simple : on calcule ce temps pour une série de courbes différentes.

Soient $A(0,0)$ et $B(-L, L)$. On considère quatre courbes différentes :

1. $y_1(x)$ consiste en l'axe des y pour $0 > y \geq -L$ suivi par l'axe des x pour $0 < x < L$.
2. $y_2(x)$ consiste en l'axe des y pour $0 > y \geq -L/2$ suivi par la droite $y = -1/2(x + L)$ pour $0 < x \leq L$.
3. $y_3(x)$ est la droite $y_2 = -2x$ pour $0 < y \leq L/2$ suivi par l'axe des x pour $L/2 < x \leq L$.
4. $y_4(x) = -x$ pour $0 \leq x \leq L$.

1. Dessiner les quatre courbes.
2. Utiliser la conservation d'énergie totale pour exprimer la vitesse v de la particule en une position (x, y) quelconque en fonction de g et y.

3. Écrire la fonctionelle $T[x]$, le temps nécessaire à la masser pour aller de A en B, en forme intégrale (soit $T[x] = \int_{y_i}^{y_f} f(x, \mathrm{d}x/\mathrm{d}y, y)\mathrm{d}y$, soit $T[y] = \int_{y_i}^{y_f} f(x, \mathrm{d}y/\mathrm{d}x, y)\mathrm{d}x$).
4. Calculer T pour les quatre chemins donnés ci-dessus. On suppose que l'énergie est toujours conservée et qu'il n'y a pas de chocs.
5. Quel chemin a le plus petit T ?

17.3 Rayons courbes et mirages

Chapter 18

Partiel 2008

18.1 Oscillateur simple

1. The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}K(r-\ell)^2 + mgr\cos\theta$$

2. The equations are

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 + mg\cos\theta - K(r-\ell) \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} &= -gr\sin\theta \end{aligned}$$

3. Since $\partial_t L = 0$, the corresponding conserved quantity is

$$\begin{aligned} H &= \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L \\ &= T + V \end{aligned}$$

   It is the total energy.

4. THe Lagrangian is now

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}K(r-\ell)^2$$

   A rotation by a fixed angle α takes

$$\begin{aligned} \theta &\longrightarrow \theta + \alpha \\ \dot{\theta} &\longrightarrow \dot{\theta} \\ L &\longrightarrow L \end{aligned}$$

   The angular momentum $mr^2\dot{\theta}$ is conserved.

5. The equation for r is

$$\ddot{r} = \frac{J^2}{m^2r^3} - \frac{K}{m}(r-\ell)$$

   Initial conditions are

$$r(0) = r_0 \qquad \dot{r}(0) = 0 \qquad \dot{\theta}(0) = 0$$

Since $J = mr^2\dot{\theta}$ is time independent, $J = 0$.
Let $z = r - \ell$, then

$$\ddot{z} = -\frac{K}{m}\,z$$
$$\Longrightarrow z = A\cos\left(\sqrt{\frac{K}{m}}t\right) + B\sin\left(\sqrt{\frac{K}{m}}t\right)$$
$$\Longrightarrow r = \ell + A\cos\left(\sqrt{\frac{K}{m}}t\right) + B\sin\left(\sqrt{\frac{K}{m}}t\right)$$

We have $B = 0$ and $A = r_0 - \ell$, so

$$r = \ell + (r_0 - \ell)\cos\left(\sqrt{\frac{K}{m}}t\right)$$

18.2 Oscillateurs couplés

1. We have

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$
$$V = \frac{1}{2}K(x_1 - \ell)^2 + \frac{1}{2}\lambda(x_2 - x_1 - \ell)^2 + \frac{1}{2}K(x_2 - 2\ell)^2$$

so

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}K(x_1 - \ell)^2 + \frac{1}{2}\lambda(x_2 - x_1 - \ell)^2 + \frac{1}{2}K(x_2 - 2\ell)^2$$

2. Let $\sqrt{2}X = x_1 + x_2$ and $\sqrt{2}Y = x_1 - x_2$, then

$$x_1 = \frac{X+Y}{\sqrt{2}} \qquad x_2 = \frac{X-Y}{\sqrt{2}}$$

and

$$\dot{x}_1^2 + \dot{x}_2^2 = \dot{X}^2 + \dot{Y}^2$$
$$(x_1 - \ell)^2 = (x_1^2 - 2x_1\ell + \ell^2) = \frac{1}{2}(X^2 + Y^2 + 2XY) - \sqrt{2}(X+Y)\ell + \ell^2$$
$$(x_2 - x_1 - \ell)^2 = (-\sqrt{2}Y - \ell)^2 = 2Y^2 + 2\sqrt{2}\ell Y + \ell^2)$$
$$(x_2 - 2\ell)^2 = x_2^2 + 4\ell^2 - 4\ell x_2 = 4\ell^2 - 2\sqrt{2}\ell(X - Y) + \frac{1}{2}(X^2 + Y^2 - 2XY)$$

so

$$\begin{aligned} L =& \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) - \frac{1}{2}Y^2(K + 2\lambda) + \frac{3}{\sqrt{2}}KX \\ & - \frac{1}{2}Y\ell(\sqrt{2}K + 2\sqrt{2}\lambda) - \ell^2\left(\frac{5}{2}K + \frac{\lambda}{2}\right) \end{aligned}$$

3. Let $X = \overline{X} + a\ell$ and $Y = \overline{Y} + b\ell$, then

$$\alpha X^2 = \alpha \overline{X}^2 + \alpha 2\overline{X}a\ell + \alpha a^2\ell^2$$
$$\gamma X\ell = \gamma \overline{X}\ell + \gamma a\ell^2$$

   To remove terms from the Lagrangian which are linear in $\overline{X}$, one must choose

$$2\alpha a\ell + \gamma\ell = 0$$
$$\implies a = \frac{-\gamma}{2\alpha}$$

   and similarly

$$b = \frac{-\delta}{2\beta}$$

   So

$$L = \frac{1}{2}m(\dot{\overline{X}}^2 + \dot{\overline{Y}}^2) + \alpha\overline{X}^2 + \beta\overline{Y}^2$$

4. Euler–Lagrange for $\overline{X}$ and $\overline{Y}$ are

$$\ddot{\overline{X}} + \omega_1^2\overline{X} = 0 \qquad \omega_1^2 = \frac{2\alpha}{m}$$
$$\ddot{\overline{Y}} + \omega_2^2\overline{Y} = 0 \qquad \omega_2^2 = \frac{2\beta}{m}$$

18.3 Gagner un slalom

1. The total energy E of the skier is a constant.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$V = -mgx\sin\alpha$$
$$\implies E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgx\sin\alpha$$

   and $E = 0$ with the initial conditions.

2. We have

$$\mathrm{d}t = \frac{\mathrm{d}\ell}{v}$$

   and

$$(\mathrm{d}t)^2 = \frac{(\mathrm{d}\ell)^2}{v^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{x^2 + y^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{2gx\sin\alpha}$$

3.

$$T = \int \mathrm{d}t = \frac{1}{\sqrt{2g\sin\alpha}} \int_0^{x_n} \underbrace{\frac{\sqrt{1 + y'^2}}{\sqrt{x}}}_{= f}\, \mathrm{d}x$$

4. The optimal trajectory must satisfy the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} = 0$$

so $\partial_{y'} = cste$.

5. Then

$$\frac{y'}{\sqrt{x(1+y'^2)}} = c$$

6. We have

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$

and

$$\dot{x} = \frac{\dot{y}}{y'} = \dot{y}\sqrt{\frac{1-c^2x}{c^2x}}$$

But $x^2 + y^2 = 2gx\sin\alpha$. Hence substituting this in the previous equation

$$\frac{\dot{y}^2}{\dot{x}^2} = 2gc^2\sin\alpha$$

Appendix

1

Appendix A

Quantum field theory: an introduction

A.1 Relativistic particle kinematics

In special relativity one have space-time 4-vector $x^\mu = (ct, \boldsymbol{x})$ and a metric $\eta_{\mu\nu}$. One can construct a new 4-vector $x_\nu = \eta_{\mu\nu} x^\mu = (ct, -\boldsymbol{x})$ where

$$\eta_{\mu\nu} = \begin{pmatrix} I & \tilde{0} \\ \tilde{0} & -I_3 \end{pmatrix} \tag{A.1}$$

We determine the particle kinematics with

$$p^\mu = m \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(\frac{E}{c}, \boldsymbol{p} \right) \tag{A.2}$$

and one can construct quantities independent of the frame in which we are moving:

$$x_\mu x^\mu = c^2 t^2 - \boldsymbol{x}^2 \tag{A.3}$$

$$\begin{aligned} p_\mu p^\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2 \\ E^2 = m^2 c^4 + p^2 c^2 \end{aligned} \tag{A.4}$$

so

$$E = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} \sim mc^2 + \frac{p^2}{2m} + \cdots \tag{A.5}$$

A.2 Non relativistic quantum mechanics

One have a wave function $\psi(\boldsymbol{x}, t)$. $|\psi(\boldsymbol{x}, t)|^2$ is a probability density. The Schrödinger equation's for a particle of mass m is

$$\hat{H}\psi = \frac{p^2}{2m}\psi \tag{A.6}$$

$$i\hbar \frac{\partial}{\partial t} \psi(\boldsymbol{x}, t) = \frac{-\hbar^2}{2m} \nabla^2 \psi(\boldsymbol{x}, t) \tag{A.7}$$

A possible way to make particle relativistic is

$$E = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \tag{A.8}$$

and

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x}, t) = mc^2\left(\sqrt{1 - \frac{\hbar^2\,\nabla^2}{m^2c^2}}\right)\psi(\boldsymbol{x}, t) \tag{A.9}$$

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(\boldsymbol{x}, t) = m^2c^4\left(1 - \frac{\hbar^2\,\nabla^2}{m^2c^2}\right) \tag{A.10}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \tag{A.11}$$

We use the notation

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \partial_\mu\partial^\mu \tag{A.12}$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} \tag{A.13}$$

If we put a particle in a box, we have

$$\Delta x \Delta p \sim \hbar \tag{A.14}$$

so

$$p \longrightarrow p + (p + \overline{p}) \tag{A.15}$$

A.3 Simple harmonic oscillator

We work yet in natural units with

$$c = \hbar = 1 \tag{A.16}$$

and the units change to

$$\begin{gathered}[\text{momentum}] = [\text{energy}]\\ [\text{time}] = [\text{space}]\\ [\text{space-time}] = [\text{energy-momentum}]\end{gathered}$$

so

$$(\Box + m^2)\psi = 0 \tag{A.17}$$

Let be $m = 1$ and the hamiltonian become

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{1}{2}\omega\hat{x}^2 \tag{A.18}$$

and we look for energy eigenstates

$$\hat{H}\left|E\right\rangle = E\left|E\right\rangle \tag{A.19}$$

We have

$$\langle x|\hat{H}|E\rangle = E\underbrace{\langle x|E\rangle}_{=\psi_E(\boldsymbol{x},t)} \tag{A.20}$$

The commutator between $\hat{p}$ and $\hat{x}$ is defined by

$$[\hat{x},\hat{p}] = i \tag{A.21}$$

We define

$$\hat{a} = \frac{\omega\hat{x}+i\hat{p}}{\sqrt{2\omega}} \tag{A.22}$$

$$\hat{a}^\dagger = \frac{\omega\hat{x}-i\hat{p}}{\sqrt{2\omega}} \tag{A.23}$$

and so

$$\hat{H} = \frac{1}{2}\omega(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger) = \omega\hat{a}^\dagger\hat{a}+\frac{1}{2}\omega \tag{A.24}$$

The commutator between $\hat{a}^\dagger$ and $\hat{a}$ is

$$\begin{aligned}\left[\hat{a},\hat{a}^\dagger\right] &= \frac{1}{2\omega}\left[\omega\hat{x}+i\hat{p},\omega\hat{x}-i\hat{p}\right] \\ &= \frac{1}{2\omega}\left([\omega\hat{x},-i\hat{p}]+[i\hat{p},\omega\hat{x}]\right) = 1\end{aligned}$$

We denote the ground state of the simple harmonic oscillator by $|0\rangle$ and

$$H|0\rangle = \frac{1}{2}\omega|0\rangle \qquad \hat{a}|0\rangle = 0 \tag{A.25}$$

and since

$$\left[\hat{H},\hat{a}^\dagger\right] = \omega\hat{a}^\dagger \tag{A.26}$$

we find

$$\hat{H}(\hat{a}^\dagger|0\rangle) = \underbrace{\hat{a}^\dagger\hat{H}|0\rangle}_{=\frac{1}{2}\omega(\hat{a}^\dagger|0\rangle)} + \omega(\hat{a}^\dagger|0\rangle) \tag{A.27}$$

Let $|1\rangle = \hat{a}^\dagger|0\rangle$ so

$$\hat{H}|1\rangle = \left(\omega+\frac{1}{2}\omega\right)|1\rangle \tag{A.28}$$

A.4 Relativistic quantum mechanical particles

Particles with only mass m:
– bosons;
– non-interacting
are put in a square box of length L.

We have $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ with

$$\boldsymbol{k} = \frac{2\pi}{\omega}\boldsymbol{x} \tag{A.29}$$

and

$$e^{ik_1L} = e^{ik_2L} = e^{ik_3L} = 1 \tag{A.30}$$

The ground state (no particles — vacuum) is denoted by $|0\rangle$.

We define the operator $\hat{a}^\dagger(\boldsymbol{k})$, which creates a particle of momemtum $\boldsymbol{k}$ and hence

$$\omega_k = \sqrt{m^2 + \boldsymbol{k}^2} \tag{A.31}$$

so

$$\hat{a}^\dagger(\boldsymbol{k}) |0\rangle = |\boldsymbol{k}\rangle \tag{A.32}$$

and

$$\left.\begin{aligned} \hat{a}^\dagger(\boldsymbol{k_1})\hat{a}^\dagger(\boldsymbol{k_2}) |0\rangle = |\boldsymbol{k_1} \cdot \boldsymbol{k_2}\rangle \\ \hat{a}^\dagger(\boldsymbol{k_2})\hat{a}^\dagger(\boldsymbol{k_1}) |0\rangle = |\boldsymbol{k_2} \cdot \boldsymbol{k_1}\rangle \end{aligned}\right\} \implies \left[\hat{a}^\dagger(\boldsymbol{k_1}), \hat{a}^\dagger(\boldsymbol{k_2})\right] = 0 \tag{A.33}$$

since $|\boldsymbol{k_1} \cdot \boldsymbol{k_2}\rangle = |\boldsymbol{k_2} \cdot \boldsymbol{k_1}\rangle$ because it is bosons [1]: we don't know which is which.

$\hat{a}(\boldsymbol{k})$ removes a particle of momemtum $\boldsymbol{k}$ from the box, so

$$\hat{a} |0\rangle = 0 \qquad \hat{a}(\boldsymbol{k}) |\boldsymbol{k}\rangle = |0\rangle \tag{A.34}$$

hence

$$\begin{aligned} \left[\hat{a}(\boldsymbol{k}), \hat{a}(\boldsymbol{k'})\right] &= 0 \\ \left[\hat{a}(\boldsymbol{k}), \hat{a}^\dagger(\boldsymbol{k'})\right] &= \delta_{\boldsymbol{kk'}} \end{aligned}$$

We want an Hamiltonian $\hat{H}_0$

$$\begin{aligned} \hat{H}_0 |0\rangle &= 0 \\ \hat{H}_0 |\boldsymbol{k}\rangle &= \underbrace{\sqrt{\omega^2 + \boldsymbol{k}^2}}_{=\omega_k} |\boldsymbol{k}\rangle \end{aligned}$$

We can write

$$\hat{H}_0 |\boldsymbol{k_1}, \boldsymbol{k_2}\rangle = (\omega_{k_1} + \omega_{k_2}) |\boldsymbol{k_1}, \boldsymbol{k_2}\rangle \tag{A.35}$$

so

$$\boxed{\hat{H}_0 = \sum_{\boldsymbol{k}} \omega_k \hat{a}^\dagger(\boldsymbol{k})\hat{a}(\boldsymbol{k})} \tag{A.36}$$

And

$$\begin{aligned} \hat{H}_0 |\boldsymbol{p}\rangle &= \sum \omega_k \hat{a}^\dagger(\boldsymbol{k}) \underbrace{\hat{a}(\boldsymbol{k})\hat{a}^\dagger(\boldsymbol{p})}_{=\delta_{\boldsymbol{kp}}} |0\rangle \\ &= \sum \omega_k \delta_{\boldsymbol{kp}} \hat{a}^\dagger(\boldsymbol{k}) |0\rangle \\ &= \omega_p \hat{a}(\boldsymbol{p}) |0\rangle \\ &= \omega_p |\boldsymbol{k}\rangle \end{aligned}$$

We deduce $\hat{H}$ from $\hat{H}_0$:

$$\begin{aligned} \hat{H} &= \sum \frac{1}{2}\omega_k \Big(\hat{a}^\dagger(\boldsymbol{k})\hat{a}(\boldsymbol{k}) + \hat{a}(\boldsymbol{k})\hat{a}^\dagger(\boldsymbol{k})\Big) \\ &= H_0 + \sum_{\boldsymbol{k}}^{\Lambda} \frac{1}{2}\omega_k \sim \Lambda^4 \end{aligned}$$

1. which are indistinguishible.

Remarque : The operators all depend of time.

The Heisenberg's equations are

$$\left[\hat{a}(\boldsymbol{k},t),\hat{H}_0\right] = i\dot{\hat{a}}(\boldsymbol{k},t) \tag{A.37}$$

and

$$\begin{aligned}\hat{a}(\boldsymbol{k},t) &= \hat{a}(\boldsymbol{k})\,\mathrm{e}^{-i\omega_k t}\\ \hat{a}^\dagger(\boldsymbol{k},t) &= \hat{a}(\boldsymbol{k})\,\mathrm{e}^{i\omega_k t}\end{aligned}$$

so

$$\begin{aligned}\hat{a}(\boldsymbol{k},t) &= \frac{\omega_k\hat{\varphi}(\boldsymbol{k},t)+i\hat{\pi}(\boldsymbol{k},t)}{\sqrt{2\omega}}\\ \hat{a}^\dagger(\boldsymbol{k},t) &= \frac{\omega_k\hat{\varphi}(\boldsymbol{k},t)-i\hat{\pi}(\boldsymbol{k},t)}{\sqrt{2\omega}}\end{aligned}$$

and we have

$$\begin{aligned}\hat{\varphi}(\boldsymbol{k},t) &= \sum_{\boldsymbol{k}}\frac{1}{\sqrt{V}}\hat{\varphi}(\boldsymbol{k},t)\,\mathrm{e}^{\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \sum\frac{1}{\sqrt{2\omega_k V}}\left(\hat{a}(\boldsymbol{k})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega_k t}+\hat{a}^\dagger(\boldsymbol{k})\,\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_k t}\right)\\ &= \sum\frac{1}{\sqrt{2\omega_k V}}\left(\hat{a}(\boldsymbol{k})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}+\hat{a}^\dagger(\boldsymbol{k})\,\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)\end{aligned}$$

where V is the volume of the box. So

$$k = |\omega,\boldsymbol{k}\rangle \tag{A.38}$$

$$x = |t,\boldsymbol{x}\rangle \tag{A.39}$$

Remarque : We have

$$\begin{aligned}\hat{\varphi}^\dagger(\boldsymbol{k},t) &= \hat{\varphi}(\boldsymbol{x},t)\\ \hat{\varphi}^\dagger(\boldsymbol{k},t) &= \hat{\varphi}(-\boldsymbol{k},t)\qquad\text{non charge}\end{aligned}$$

And hence, with (A.17)

$$\begin{aligned}(\Box+m^2)\,\mathrm{e}^{\pm ik\cdot x} = \pm\left(\omega_k^2-(k^2+m^2)\right)\mathrm{e}^{\pm ik\cdot x} = 0\\ \Longrightarrow (\Box+m^2)\hat{\varphi}(t,\boldsymbol{x}) = 0\end{aligned}$$

We can also write the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)-\frac{1}{2}m^2\varphi^2 \tag{A.40}$$

and the action is

$$S = \int\mathcal{L}\,\mathrm{d}x\mathrm{d}t \tag{A.41}$$

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